3d quantum trace map

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This talk is about my recent joint work with Sam Panitch (arXiv:2403.12850) constructing the 3d quantum trace map

$$\mathsf{Tr}_{\mathcal{T}} : \mathrm{Sk}(Y) \to \mathrm{SQGM}_{\mathcal{T}}(Y)$$

that relates two quantizations of the $\mathrm{SL}_2(\mathbb{C})$ character variety of a 3-manifold Y equipped with an ideal triangulation $\mathcal{T}$.

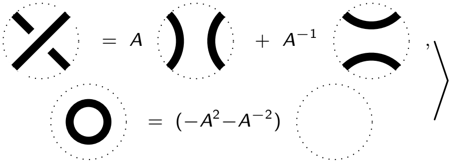
1. Background

Skein module $\mathrm{Sk}(Y)$

Definition

Let Y be an oriented 3-manifold, and let $R = \mathbb{Z}[A^{\pm 1}]$ or $\mathbb{Q}(A)$ be our base ring.
The (Kauffmann bracket / SL_2) skein module of Y is

$$\mathrm{Sk}(Y) := \frac{R\langle\text{isotopy classes of framed, unoriented links in } Y\rangle}{\left\langle \text{skein rels:} \right\rangle}$$



Elements of $\mathrm{Sk}(Y)$ are called skeins.

Some properties of $\mathrm{Sk}(Y)$

- $\mathrm{Sk}(S^3) = R[\emptyset]$, i.e., it is free of rank 1, spanned by the empty skein $[\emptyset]$;
$$[K] = \langle K \rangle [\emptyset], \quad \langle K \rangle \in R.$$
- When $Y = \Sigma \times I$, it is a unital associative algebra
$$\mathrm{SkAlg}(\Sigma) := \mathrm{Sk}(\Sigma \times I)$$
called the skein algebra of Σ.
- For each boundary component $\Sigma \subset \partial Y$, $\mathrm{Sk}(Y)$ is a module over $\mathrm{SkAlg}(\Sigma)$.

Some properties of $\mathrm{Sk}(Y)$

- If we specialize $A = \pm 1$, $\mathrm{Sk}_{\pm 1}(Y)$ becomes a commutative algebra. In fact, $\mathrm{Sk}_{-1}(Y)$ is canonically isomorphic to the coordinate ring of $\mathrm{SL}_2(\mathbb{C})$-character variety of Y [Bullock (1997)], [Przytycki, Sikora (2000)]:

  $$\begin{aligned} \mathrm{Sk}_{-1}(Y) &\xrightarrow{\sim} \mathbb{C}[\mathcal{X}^{\mathrm{SL}_2(\mathbb{C})}(Y)] \\ \gamma &\mapsto (\rho \mapsto -\mathrm{Tr}(\rho(\gamma))) \end{aligned}$$

  where

  $$\mathcal{X}^{\mathrm{SL}_2(\mathbb{C})}(Y) := \mathrm{Hom}(\pi_1(Y), \mathrm{SL}_2(\mathbb{C})) /\!\!/ \mathrm{SL}_2(\mathbb{C}).$$

- For a closed 3-manifold Y, $\mathrm{Sk}(Y)$ is finite dimensional over $\mathbb{Q}(A)$ for generic A [Gunningham, Jordan, Safronov (2019)].

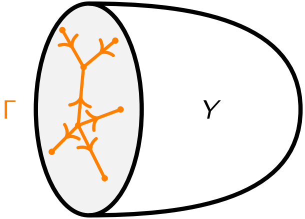
Relative version of $\mathrm{Sk}(Y)$

We can study skein modules locally by cutting Y into simpler pieces! A relative version of skein module, known as the stated skein module has been studied by [Bonahon, Wong (2010)], [Lê (2016)], [Costantino, Lê (2019)], ...

Here, we use a slightly generalized definition as used in [Panitch, P. (2024)]:

Definition

Let Y be an oriented 3-manifold and $\Gamma \subset \partial Y$ an oriented (bipartite) graph, each of whose vertices are either a source or a sink.

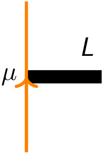


Γ will be called a boundary marking of Y.

Relative version of $\mathrm{Sk}(Y)$

Definition (cont.)

A stated tangle in (Y, Γ) is an unoriented ribbon tangle L in Y with boundary lying flat in $\Gamma \setminus V(\Gamma)$, equipped with a function (called a state) $\partial L \to \{\pm\}$; i.e., each end point of a stated tangle looks like



$$, \quad \mu \in \{\pm\}.$$

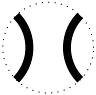
Let $R = \mathbb{Z}[A^{\pm\frac{1}{2}}, (-A^2)^{\frac{1}{2}}]$ or $\mathbb{Q}(A^{\frac{1}{2}}, (-1)^{\frac{1}{2}})$ be our base ring.
The (stated) skein module $\mathrm{Sk}(Y, \Gamma)$ is

$$\mathrm{Sk}(Y, \Gamma) := \frac{R\langle \text{isotopy classes of stated tangles in } (Y, \Gamma) \rangle}{\langle \text{stated skein relations} \rangle},$$

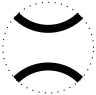
where the stated skein relations are given in the next slide:

Relative version of $\mathrm{Sk}(Y)$

$= A$



$+ \; A^{-1}$



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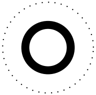
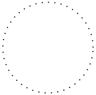


$= \; (-A^2 - A^{-2})$



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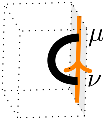
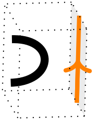
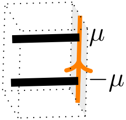


$= \; \delta_{\mu,-\nu}\,(-A^2)^{\frac{\mu}{2}} \,, \qquad \mu,\nu \in \{\pm 1\} \,,$



$= \; \sum_{\mu\in\{\pm\}} (-A^2)^{\frac{\mu}{2}}$



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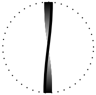
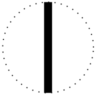


$= \; (-A^3)^{\frac{1}{2}}$



$.$

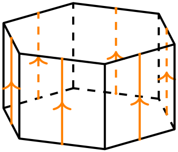
Relative version of $\mathrm{SkAlg}(\Sigma)$

For an oriented surface Σ whose boundary consists of intervals (e.g., an n-gon D_n, which is a disk D^2 with n punctures on the boundary), we can choose a set of marked points $P \subset \partial\Sigma$, consisting of one point from each interval.

Then, the stated skein algebra

$$\mathrm{SkAlg}(\Sigma) := \mathrm{Sk}(\Sigma \times I, P \times I)$$

is a unital associatve algebra.

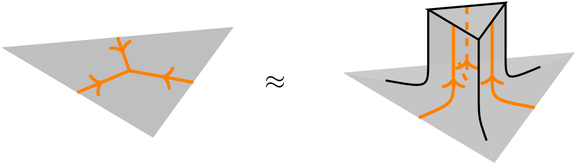


Bimodule structure on $\mathrm{Sk}(Y, \Gamma)$

The first key idea:

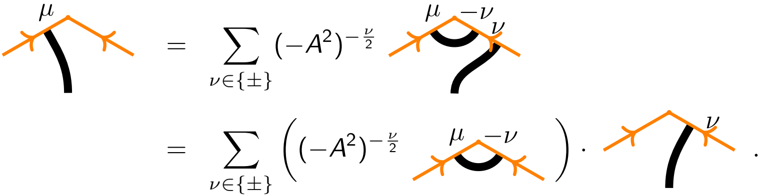
Proposition ([Panitch, P. (2024)])

$\mathrm{Sk}(Y, \Gamma)$ has a natural $\otimes_{v \in V(\Gamma)^+} \mathrm{SkAlg}(D_{\deg v})$-$\otimes_{w \in V(\Gamma)^-} \mathrm{SkAlg}(D_{\deg w})$-bimodule structure, where $V(\Gamma)^+$ (resp. $V(\Gamma)^-$) is the set of sink (resp. source) vertices of Γ.



Bimodule structure on $\mathrm{Sk}(Y, \Gamma)$

Considering these skein algebra actions as coefficients, we get “skein relations” associated to sliding an end point of a tangle across a vertex of the boundary marking Γ, e.g.,



$$= \sum_{\nu\in\{\pm\}} (-A^2)^{-\frac{\nu}{2}}$$

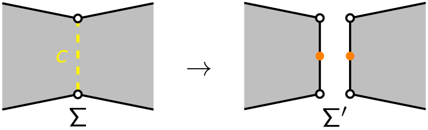
$$= \sum_{\nu\in\{\pm\}} \Bigg((-A^2)^{-\frac{\nu}{2}} \Bigg) \cdot \quad .$$

This bimodule structure will play an important role later. For now, let’s review the story in 2d.

Splitting homomorphism

Theorem ([Lê (2016)], [Costantino, Lê (2019)])

Let c be an ideal arc in Σ and Σ' the surface obtained by cutting Σ along c.



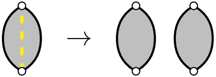
Then, there is an algebra homomorphism (called the splitting homomorphism)

$$\sigma_c : \mathrm{SkAlg}(\Sigma) \rightarrow \mathrm{SkAlg}(\Sigma')$$
$$[L] \mapsto \sum_{\epsilon} [L'_\epsilon],$$

where ϵ ranges over $\{\pm\}^{L \cap c}$, and L'_ϵ denotes the stated tangle obtained by cutting L along $c \times I$ and assigning the state ϵ to the new ends.

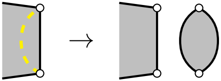
Some properties of $\mathrm{SkAlg}(\Sigma)$

- The splitting map makes $\mathrm{SkAlg}(D_2)$ a coalgebra.



  In fact, $\mathrm{SkAlg}(D_2) \cong \mathcal{O}_{q=A^2}(\mathrm{SL}_2)$ as Hopf algebras. [Costantino, Lê (2019)]

- For each boundary interval of Σ, $\mathrm{SkAlg}(\Sigma)$ is a $\mathrm{SkAlg}(D_2)$-comodule.



- After splitting, $\mathrm{SkAlg}(\Sigma')$ is a bicomodule over $\mathrm{SkAlg}(D_2)$. [Costantino, Lê (2019)] proved that

$$\sigma_c : \mathrm{SkAlg}(\Sigma) \to HH^0(\mathrm{SkAlg}(\Sigma')) \subset \mathrm{SkAlg}(\Sigma')$$

  is an algebra isomorphism.

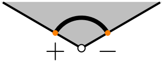
Reduced skein modules

Skein algebras/modules admit nice quotients called reduced skein algebras/modules which are often almost commutative.

Definition

A stated tangle in (Y, Γ) of the form

(seen from outside of Y) is called a bad arc.
E.g., bad arcs in $\Sigma \times I$ are the ones of the form



[Costantino, Lê (2019)]

Reduced skein modules

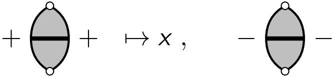
Definition (cont.)

The reduced skein algebra $\overline{\mathrm{SkAlg}}(\Sigma)$ is the quotient of $\mathrm{SkAlg}(\Sigma)$ by the two-sided ideal generated by bad arcs.
The reduced skein module $\overline{\mathrm{Sk}}(Y,\Gamma)$ is defined similarly.

Proposition ([Lê (2016)], [Costantino, Lê (2019)])

(1)

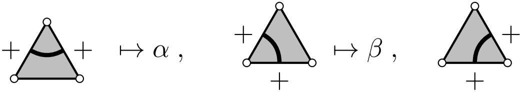
$$\overline{\mathrm{SkAlg}}(D_2) \cong \mathbb{B} := R[x^{\pm 1}]$$



$$\mapsto x\,, \qquad \mapsto x^{-1}.$$

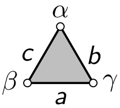
(2)

$$\overline{\mathrm{SkAlg}}(D_3) \cong \mathbb{T} := \frac{R\langle \alpha^{\pm 1}, \beta^{\pm 1}, \gamma^{\pm 1}\rangle}{\langle \beta\alpha = A\alpha\beta,\ \gamma\beta = A\beta\gamma,\ \alpha\gamma = A\gamma\alpha\rangle}$$



$$\mapsto \alpha\,, \qquad \mapsto \beta\,, \qquad \mapsto \gamma\,.$$

Extended triangle algebra



Lemma

Let

$$\widetilde{\mathbb{T}} := \frac{R\langle a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\rangle}{\langle ba = Aab,\ cb = Abc,\ ac = Aca\rangle}.$$

Then,

$$\begin{aligned} &\mathbb{T} \to \widetilde{\mathbb{T}} \\ &\alpha \mapsto [bc], \\ &\beta \mapsto [ca], \\ &\gamma \mapsto [ab] \end{aligned}$$

is an algebra embedding.

2d quantum trace map

Theorem ([Bonahon, Wong (2010)], [Costantino, Lê (2019)])

Let Σ be an ideally triangulated surface. Then, the composition

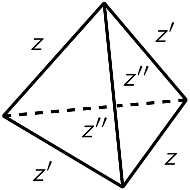
$$\overline{\mathrm{SkAlg}}(\Sigma) \xrightarrow{\sigma} \bigotimes_{f \in \mathcal{F}} \mathbb{T}_f \hookrightarrow \bigotimes_{f \in \mathcal{F}} \widetilde{\mathbb{T}}_f$$

is an algebra homomorphism from the skein algebra to the quantum Teichmuller space (or the square root Chekhov-Fock algebra, to be more precise), thus relating the two quantizations of the character variety of surfaces.

2. Main construction

Generalization to 3d?

A 3d analog of quantum Teichmuller space is the quantum gluing module, which is a two-sided quotient of a quantum torus, consisting of Laurent polynomials in quantized shape parameters $\{\hat{z}_T, \hat{z}'_T, \hat{z}''_T\}_{T \in \mathcal{T}}$ [Dimofte (2011)], [Agarwal, Gang, Lee, Romo (2022)].



,

$$\begin{gathered}\hat{z}\hat{z}' = (-A^2)^2 \hat{z}'\hat{z}, \ \ \hat{z}'\hat{z}'' = (-A^2)^2 \hat{z}''\hat{z}', \ \ \hat{z}''\hat{z} = (-A^2)^2 \hat{z}\hat{z}'', \\ [\hat{z}\hat{z}'\hat{z}''] = -A^2, \\ \hat{z}' = 1 - \hat{z}^{-1} \ \ \text{(as left actions)}, \\ \Big[\prod\nolimits_{\text{around an edge}} \hat{z}_T^{\square}\Big] = (-A^2)^2 \ \ \text{(as right actions)}\end{gathered}$$

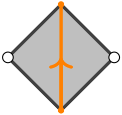
So, we would like to construct a homomorphism from the skein module to the (square root) quantum gluing module.

Combinatorially foliated surfaces

Motivated by WKB foliations in 2d and 3d spectral networks [Gaiotto, Moore, Neitzke (2012)], [Freed, Neitzke (2022)], we introduce the following notion of combinatorial foliation:

Definition ([Panitch, P. (2024)])

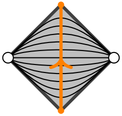
A combinatorial foliation of Σ is a decomposition of Σ into pieces, each of which is homeomorphic to the elementary quadrilateral: a quadrilateral with 2 opposite vertices removed and a diagonal marking:


.

We require that, when two edges are glued together, the orange vertices adjacent to the edges must be both sources or sinks, and that no two edges of the same elementary quadrilateral are glued together.

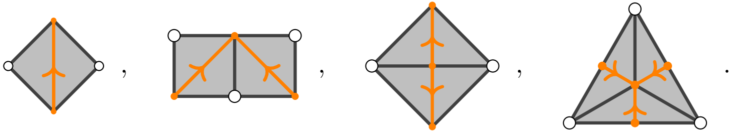
Combinatorially foliated surfaces

To each combinatorial foliation, one can associate a topological 1-dimensional foliation



whose generic leaves are in 1-1 correspondence to the interior points of the marking.

Some examples of combinatorially foliated surfaces:



Splitting homomorphism

Theorem ([Panitch, P. (2024)])

Let (Y_1, Γ_1) *and* (Y_2, Γ_2) *be boundary marked 3-manifolds. Suppose that* $\Sigma_1 \subset \partial Y_1$ *and* $\Sigma_2 \subset \partial Y_2$*, along with their markings, are homeomorphic combinatorially foliated surfaces of opposite orientations, so that we can glue* Y_1 *and* Y_2 *by identifying* Σ_1 *with* Σ_2*. Let* $Y = Y_1 \cup_\Sigma Y_2$ *be the glued 3-manifold, with boundary marking* $\Gamma = (\Gamma_1 \cup \Gamma_2) \setminus \operatorname{int} \Sigma$*.*
Then, there is a bimodule homomorphism (splitting homomorphism)

$$\begin{aligned} \sigma \ : \ \mathrm{Sk}(Y, \Gamma) &\to \mathrm{Sk}(Y_1, \Gamma_1) \mathbin{\overline{\otimes}} \mathrm{Sk}(Y_2, \Gamma_2) \\ [L] &\mapsto \left[\sum_{\epsilon \,\in\, \{\pm\}^{L \cap \Sigma}} [L_1^\epsilon] \otimes [L_2^\epsilon] \right], \end{aligned}$$

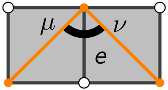
where the relative tensor product $\mathrm{Sk}(Y_1, \Gamma_1) \mathbin{\overline{\otimes}} \mathrm{Sk}(Y_2, \Gamma_2)$ *denotes the quotient of the usual tensor product* $\mathrm{Sk}(Y_1, \Gamma_1) \otimes \mathrm{Sk}(Y_2, \Gamma_2)$ *(as R-modules) by the following relations:*

Splitting homomorphism

Theorem (cont.)

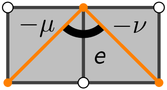
For each internal edge e of Σ*, we have the following relations:*

the action of $(-A^2)^{s\frac{\mu+\nu}{4}}$



on $\mathrm{Sk}(Y_1, \Gamma_1)$

$=$ *the action of* $(-A^2)^{-s\frac{\mu+\nu}{4}}$

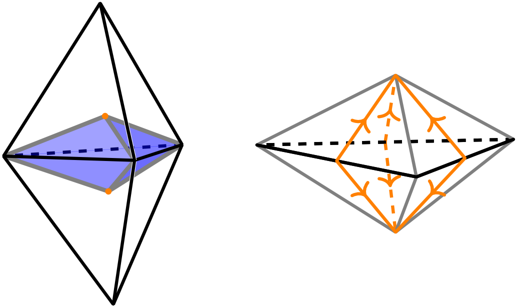


on $\mathrm{Sk}(Y_2, \Gamma_2)$,

where $s = +1$ *(resp.* -1*) if e is adjacent to a sink (resp. a source) vertex.*

Splitting into face suspensions

Let $\mathcal{T}$ be an ideal triangulation of Y. For each face $f \in \mathcal{F}$ of the ideal triangulation, its suspension Sf is the join of f with the barycenters of the two adjacent tetrahedra:



Splitting into face suspensions

Corollary ([Panitch, P. (2024)])

Let $Y = \cup_{f \in \mathcal{F}} Sf$ *be a decomposition of an ideally triangulated 3-manifold* Y *(without boundary except for cusps at infinity) into face suspensions. Then, there is a splitting map*

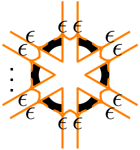
$$\overline{\sigma} \ : \ \overline{\mathrm{Sk}}(Y) \to \overline{\bigotimes}_{f \in \mathcal{F}} \overline{\mathrm{Sk}}(Sf),$$

$$[L] \mapsto \left[\sum_{\epsilon \in \left\{ \substack{\text{compatible} \\ \text{states}} \right\}} \otimes_{f \in \mathcal{F}} [L_f^{\epsilon}] \right],$$

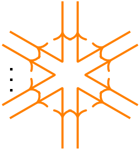
where $\overline{\bigotimes}_{f \in \mathcal{F}} \overline{\mathrm{Sk}}(Sf)$ *denotes the quotient of the usual tensor product* $\bigotimes_{f \in \mathcal{F}} \overline{\mathrm{Sk}}(Sf)$ *(as R-modules) by the following relations:*

Splitting into face suspensions

- For each internal edge $e \in \mathcal{E}$ we have the following relations among right actions on $\bigotimes_{f \in \mathcal{F}} \overline{\mathrm{Sk}}(Sf)$:

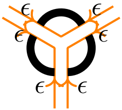


$= (-A^2)^{\epsilon}$

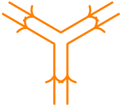


$, \quad \epsilon \in \{\pm\}.$

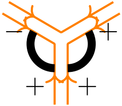
- Around each vertex cone, we have the following relations among left actions on $\bigotimes_{f \in \mathcal{F}} \overline{\mathrm{Sk}}(Sf)$: (viewed from that vertex)



$= (-A^2)^{-\epsilon}$



$, \quad \epsilon \in \{\pm\},$



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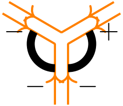

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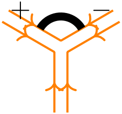


$= 0 \quad .$

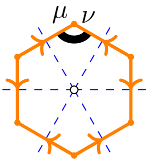
Skein modules of 3-balls

Theorem ([Panitch, P. (2024)])

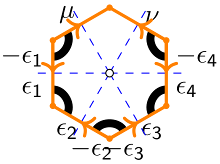
Let B be a 3-ball whose boundary is combinatorially foliated with associated marking $\Gamma \subset \partial B = S^2$. Then,

$$\mathrm{Sk}(B,\Gamma) \cong \frac{\bigotimes_{v\in V(\Gamma)^+} \mathrm{SkAlg}(D_{\mathsf{deg}\,v}) \otimes \bigotimes_{w\in V(\Gamma)^-} \mathrm{SkAlg}(D_{\mathsf{deg}\,w})^{\mathrm{op}}}{\mathrm{Ann}([\emptyset])},$$

where $\mathrm{Ann}([\emptyset])$ is the left ideal generated by the following relations, for each puncture of the combinatorial foliation of ∂B:

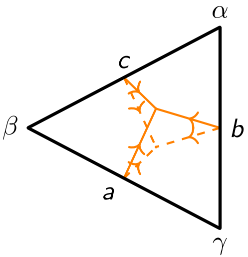


$$= \sum_{\epsilon_i\in\{\pm\}} (-A^2)^{\frac{\sum_i \epsilon_i}{2}}$$



$$, \quad \mu,\nu \in \{\pm\}.$$

Reduced skein modules of face suspensions



Corollary ([Panitch, P. (2024)])

The reduced stated skein module of a face suspension is given by

$$\overline{\mathrm{Sk}}(Sf) \cong \frac{\mathbb{T}^{\otimes 2} \otimes \mathbb{B}^{\otimes 3}}{\langle(-A^2)\alpha_1\alpha_2 = x_b x_c,\ (-A^2)\beta_1\beta_2 = x_c x_a,\ (-A^2)\gamma_1\gamma_2 = x_a x_b\rangle}$$

as a left $\mathbb{T}^{\otimes 2} \otimes (\mathbb{B}^{\mathrm{op}})^{\otimes 3} = \mathbb{T}^{\otimes 2} \otimes \mathbb{B}^{\otimes 3}$*-module.*

Quantum trace map for a face suspension

Lemma

We have the following embeddings of algebras

$$\mathbb{T}^{\otimes 2} \hookrightarrow \widetilde{\mathbb{T}}^{\otimes 2}$$

$$\begin{aligned} \alpha_1 &\mapsto A^{-1}[b_1 c_1], \quad \beta_1 \mapsto A^{-1}[c_1 a_1], \quad \gamma_1 \mapsto A^{-1}[a_1 b_1], \\ \alpha_2 &\mapsto A^{-1}[b_2 c_2], \quad \beta_2 \mapsto A^{-1}[c_2 a_2], \quad \gamma_2 \mapsto A^{-1}[a_2 b_2], \end{aligned}$$

and

$$\mathbb{B}^{\otimes 3} \hookrightarrow \widetilde{\mathbb{T}}^{\otimes 2}$$

$$x_a \mapsto (-1)^{-\frac{1}{2}} a_1 \otimes a_2, \quad x_b \mapsto (-1)^{-\frac{1}{2}} b_1 \otimes b_2, \quad x_c \mapsto (-1)^{-\frac{1}{2}} c_1 \otimes c_2.$$

Theorem ([Panitch, P. (2024)])

There is a $\mathbb{T}^{\otimes 2} - \mathbb{B}^{\otimes 3}$-bimodule embedding

$$\mathrm{Tr}_{Sf} : \overline{\mathrm{Sk}}(Sf) \to \widetilde{\mathbb{T}}^{\otimes 2} =: \mathbb{S}f$$

$$[\emptyset] \mapsto 1$$

Putting everything together

Putting everything together, we have

$$\begin{tikzcd}
 & \bigotimes_{f\in\mathcal{F}} \overline{\mathrm{Sk}}(Sf) \arrow[r, "\otimes \mathsf{Tr}_{Sf}"] \arrow[d] & \bigotimes_{f\in\mathcal{F}} \mathbb{S}f \arrow[d] \\
\mathrm{Sk}(Y) = \overline{\mathrm{Sk}}(Y) \arrow[r, "\overline{\sigma}"] \arrow[rr, dashed, bend right, "\mathsf{Tr}_{\mathcal{T}} := \overline{\otimes}\,\mathsf{Tr}_{Sf} \circ \overline{\sigma}"'] & \overline{\bigotimes}_{f\in\mathcal{F}} \overline{\mathrm{Sk}}(Sf) \arrow[r, dashed, "\overline{\otimes}\,\mathsf{Tr}_{Sf}"] & \overline{\bigotimes}_{f\in\mathcal{F}} \mathbb{S}f
\end{tikzcd},$$

where the vertical arrows are quotients, and $\overline{\bigotimes}_{f\in\mathcal{F}}\mathbb{S}f$ denotes the quotient of $\bigotimes_{f\in\mathcal{F}} \mathbb{S}f$ by the images of the relations we have in $\overline{\bigotimes}_{f\in\mathcal{F}}\overline{\mathrm{Sk}}(Sf)$.

The composition $\mathsf{Tr}_{\mathcal{T}} := \overline{\otimes}_{f\in\mathcal{F}}\,\mathsf{Tr}_{Sf} \circ \overline{\sigma}$ is the 3d quantum trace map [Panitch, P. (2024)].

Putting everything together

The image of the 3d quantum trace map

$$\mathrm{Tr}_{\mathcal{T}} : \mathrm{Sk}(Y) \to \overline{\bigotimes}_{f \in \mathcal{F}} \overline{\mathrm{Sk}}(Sf)$$

lies in the square root quantum gluing module

$$\mathrm{SQGM}_{\mathcal{T}}(Y) \subset \overline{\bigotimes}_{f \in \mathcal{F}} \overline{\mathrm{Sk}}(Sf),$$

which consists of Laurent polynomials in square roots of the quantized shape parameters.

The three types of relations we had to impose correspond exactly to the gluing relations, the vertex relations, and the Lagrangian relations among quantized shape parameters (c.f. [Dimofte (2011)]).

Summary

- We generalized the notion of stated skein modules by allowing the boundary markings to be bipartite graphs (rather than just disjoint union of intervals). We observed that $\mathrm{Sk}(Y, \Gamma)$ has a natural bimodule structure.
- We introduced the notion of combinatorial foliation, and showed the existence of splitting homomorphisms when we split 3-manifolds along combinatorially foliated surfaces.
- We proved a structure theorem for skein modules of 3-balls, and used it to construct the quantum trace map Tr_{Sf} for a face suspension Sf.
- By splitting into face suspensions and applying $\otimes \mathrm{Tr}_{Sf}$, we constructed a map from $\mathrm{Sk}(Y)$ to the square root quantum gluing module.