

# BRANCHES, QUIVERS, AND IDEALS FOR KNOT COMPLEMENTS

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## Abstract

We generalize the  $F_K$  invariant, i.e.  $\widehat{Z}$  for the complement of a knot  $K$  in the 3-sphere, the knots-quivers correspondence, and  $A$ -polynomials of knots, and find several interconnections between them. We associate an  $F_K$  invariant to any branch of the  $A$ -polynomial of  $K$  and we work out explicit expressions for several simple knots. We show that these  $F_K$  invariants can be written in the form of a quiver generating series, in analogy with the knots-quivers correspondence. We discuss various methods to obtain such quiver representations, among others using  $R$ -matrices. We generalize the quantum  $a$ -deformed  $A$ -polynomial to an ideal that contains the recursion relation in the group rank, i.e. in the parameter  $a$ , and describe its classical limit in terms of the Coulomb branch of a 3d-5d theory. We also provide  $t$ -deformed versions. Furthermore, we study how the quiver formulation for closed 3-manifolds obtained by surgery leads to the superpotential of 3d  $\mathcal{N} = 2$  theory  $T[M_3]$  and to the data of the associated modular tensor category  $\text{MTC}[M_3]$ .

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## 1. INTRODUCTION

The interface of high energy theoretical physics and knot theory (and, more generally, low dimensional topology) is interesting from several points of view. On the one hand, it yields exact and non-perturbative results that illustrate important physical concepts; on the other, it leads to definitions of new invariants of knots (and three-manifolds) or reveals relations between such invariants originally defined in other ways. In recent years, several interesting results along these lines have been found. The following examples are relevant to this paper: a 3d-3d duality that relates three-manifolds to quantum field theories with extended supersymmetry [DGG14],  $\widehat{Z}$  invariants of three-manifolds [GPV17, GPPV20], and the knots-quivers correspondence [KRSS17, KRSS19]. Our aim here is to generalize and unify these concepts.

The  $A$ -polynomial of a knot  $K$  [CCG<sup>+</sup>94, Gar04, Guk05]  $A_K(x, y)$  is an important starting point for the study undertaken in this paper. The zero set of the  $A$ -polynomial is an algebraic curve in  $(\mathbb{C}^*)^2$  which can be interpreted as the moduli space of vacua of the Chern-Simons theory with  $SL(2, \mathbb{C})$  gauge group on the complement of  $K$  in  $S^3$ . It has been known for some time that the Chern-Simons theory associates a perturbative series in  $\hbar$  (i.e., roughly, the inverse of the level  $k$ ) to various branches of the  $A$ -polynomial [DGLZ09]. All such series are annihilated by the quantum counterpart of the  $A$ -polynomial, i.e. an operator  $\widehat{A}_K(\hat{x}, \hat{y}, q)$ , with  $\hat{y}\hat{x} = q\hat{x}\hat{y}$ , and with  $q \rightarrow 1$  limit equal to the original  $A$ -polynomial. This quantum  $A$ -polynomial also gives the recursion relations (in colour) for coloured Jones polynomials. Furthermore, quite recently it was shown that the perturbative series associated to the abelian branch can be resummed into a power series  $F_K(x, q)$  in variables  $q = e^{\hbar}$  and  $x$  with integer coefficients [GM21]. The series  $F_K(x, q)$ , which is also referred to simply as the  $F_K$  invariant, can be interpreted as the  $\widehat{Z}$  invariant of the three-manifold that is the complement of  $K$  in  $S^3$ ;  $\widehat{Z}$  invariants were originally introduced for closed three-manifolds in [GPV17, GPPV20], and then generalized to the open version just mentioned.

The  $A$ -polynomial and associated invariants admit  $a$ - and  $t$ -deformations. Here, the  $a$ -deformation amounts to generalizing the  $SL(2, \mathbb{C})$  gauge group of the Chern-Simons theory to  $SL(N, \mathbb{C})$  and then replacing  $N$ -dependence of various quantities by a uniform dependence on  $a = q^N$ , while  $t$ -deformation makes contact with homological knot invariants. The  $a$ -deformed  $A$ -polynomials introduced in [AV12] are closely related to augmentation polynomials of knot contact homology [AENV14], and their further  $t$ -deformation gives rise to super- $A$ -polynomials [AGSF12, FGS13, FGSS13]. Recently, an  $SL(N, \mathbb{C})$  generalization and subsequent  $a$ -deformation of  $F_K$  invariants were proposed respectively in [Par20a] and [EGG<sup>+</sup>20].

In this paper we generalize the ( $a$ -deformed)  $F_K$  invariants to other branches  $\alpha$  of  $A$ -polynomials: we show that  $\hbar$  expansions associated to all (not only abelian) branches  $\alpha$  of  $A$ -polynomial can be also resummed into a series  $F_K^{(\alpha)}(x, q)$  with integer coefficients in  $q$  and  $x$  (and  $a$ , in the  $a$ -deformed case; in the rest of this section we often suppress  $a$  from the notation for  $F_K$ -invariants and  $A$ -polynomials). All these series are also annihilated by the same quantum  $A$ -polynomial  $\widehat{A}_K(\hat{x}, \hat{y}, q)$ ; in other words, they can be determined by the same recursion relation encoded in  $\widehat{A}_K(\hat{x}, \hat{y}, q)$ , but with different initial condition. Furthermore, we show that the initial conditions of  $F_K^{(\alpha)}(x, q)$  for various branches of  $A$ -polynomial are encoded in slopes of the boundaries of the Newton polygon associated to the  $A$ -polynomial.

We further show that these different invariants  $F_K^{(\alpha)}(x, q)$  can be expressed in the form of quiver generating series. This statement can be regarded as an extension of the knots-quivers correspondence, introduced in [KRSS17, KRSS19] and further discussed in [PSS18, PS19, EKL20b, EKL20a, EKL21, KPSS21, SW19, SW21, LNPS20, JKL<sup>+</sup>21]. The first step towards such an extension, for the  $F_K$  invariants associated to the abelian branch for  $(2, 2p + 1)$  torus knots, was taken in [Kuc20]. In the original formulation, the knots-quivers correspondence is the statement that a generating function of coloured HOMFLY-PT polynomials can be written in the form of a quiver generating series; among others, this implies that LMOV invariants [OV00, LMV00] are expressed in terms of integer motivic Donaldson-Thomas invariants of a quiver [KS11, Efi12, MR19, FR18], which proves integrality of the former invariants. A natural setup for the original formulation of the knots-quivers correspondence is the Ooguri-Vafa configuration [OV00, LMV00], which involves branes on the conormal of a knot  $K$  – for this reason, we refer to quivers that arise in the original formulation as knot conormal quivers. Analogously, we refer to quivers that encode series  $F_K^{(\alpha)}(x, q)$  as knot complement quivers. For both conormals and complements, quiver can be interpreted in enumerative geometry with nodes corresponding to certain basic holomorphic curves and arrows as their boundary linkings. Here we will also discuss the relation between the knot conormal and knot complement, as well as their corresponding quivers. We point out that writing  $F_K^{(\alpha)}(x, q)$  as a quiver generating series shows that the whole information of  $F_K^{(\alpha)}(x, q)$  is encoded in a finite set of data: the quiver matrix, slopes of boundaries of a Newton polygon, and a few other parameters that enter a quiver change of variables. Thus, the seemingly complicated, quantum and non-perturbative information about Chern-Simons theory for various branches is actually captured by a finite number of integer coefficients.

There is also an interesting interplay between classical branches of the  $A$ -polynomial and quivers encoding the corresponding  $F_K^{(\alpha)}(x, q)$ : once we assume that a quiver form of  $F_K^{(\alpha)}(x, q)$  exists, we can determine it from the form of the classical branch (up to a finite number of terms that can be fixed from the knowledge of the first few terms of  $F_K^{(\alpha)}(x, q)$ ). In this sense, the  $F_K$  series associated to various branches and (knot complement) quivers are intimately related.

We summarize the above two points in the following conjecture.

**Conjecture 1.** *Given a knot  $K$ , let  $y^{(\alpha)}(x, a)$  be a branch of  $y$  near  $x = 0$  (or  $x = \infty$ ) of the  $a$ -deformed  $A$ -polynomial of  $K$ ,  $A_K(x, y, a)$ .*

(1) *There exists a wave function  $F_K^{(\alpha)}(x, a, q)$  associated to this branch in a sense that*

$$\langle \hat{y} \rangle := \lim_{q \rightarrow 1} \frac{F_K^{(\alpha)}(qx, a, q)}{F_K^{(\alpha)}(x, a, q)} = y^{(\alpha)}(x, a)$$

*and this wave function is annihilated by the quantum  $a$ -deformed  $A$ -polynomial  $\widehat{A}_K(\hat{x}, \hat{y}, a, q)$  (which is the same for all branches  $y^{(\alpha)}(x, a)$ )*

$$\widehat{A}_K(\hat{x}, \hat{y}, a, q) F_K^{(\alpha)}(x, a, q) = 0.$$

(2) *The wave function  $F_K^{(\alpha)}(x, a, q)$  has a quiver form, in the sense of knots-quivers correspondence [KRSS17, KRSS19, Kuc20].*

In view of importance of the above extension of the knots-quivers correspondence, we develop several techniques to determine knot complement quivers corresponding to various

branches of  $A$ -polynomial. One straightforward approach amounts to determining explicitly the quiver generating series from the first several terms in the  $F_K$  series, taking advantage of the recursion relation encoded in quantum  $A$ -polynomial (applied to relevant initial terms). Another approach that we propose is based on a redefinition of knot conormal quivers that involves a change of framing. Furthermore, taking advantage of the results of [Par20b, Par21], we show that  $F_K$  in quiver form can be reconstructed from  $R$ -matrices, as well as inverted Habiro series; in some specific examples we show that this approach works also for the  $SL(N, \mathbb{C})$  case and then generalized to include  $a$ -dependence. In general, it is not guaranteed that  $F_K$  for a given branch can be determined by all these approaches; however, combining them, we can determine  $F_K$  for various branches of  $A$ -polynomial for many knots.

We also extend the  $A$ -polynomial. As mentioned above, we consider  $F_K$  invariants for various branches of  $A$ -polynomial, which gives the algebraic curve  $A_K(x, y) = 0$  with  $x, y \in \mathbb{C}^*$ . The quantum  $A$ -polynomial  $\widehat{A}_K(\hat{x}, \hat{y})$  with  $\hat{y}\hat{x} = q\hat{x}\hat{y}$  gives the recursion relations for the series  $F_K^{(\alpha)}(x, q)$ . In the  $a$ -deformed case, there are two variables:  $x$  and  $a$ , and degenerating the brane system of the knot complement to a multiple of the unknot conormal, we see that the variables  $x$  and  $xa$  play almost identical roles. It is then natural to try to promote also the variable  $a$  to an operator  $\hat{a}$ . This is indeed possible, and after introducing a dual variable  $b$ , we find another algebraic curve  $\{B_K(a, b) = 0\}$ , which is the zero set of what we call the  $B$ -polynomial. Also the  $B$ -polynomial has a quantum counterpart  $\widehat{B}_K(\hat{a}, \hat{b})$ , with  $\hat{b}\hat{a} = q\hat{a}\hat{b}$ , which again annihilates  $F_K^{(\alpha)}(x, a, q)$ . Thus, similarly to the quantum  $A$ -polynomial, the quantum  $B$ -polynomial captures information about  $F_K^{(\alpha)}(x, a, q)$  that can be extracted recursively. Note also that  $a$  is the variable associated to the closed holomorphic central  $\mathbb{CP}^1$  in the resolved conifold and the  $B$ -polynomial indeed encodes information about closed string contributions (which are not detected by the original  $A$ -polynomial).

Note that, in case of  $SL(N, \mathbb{C})$  theory, the authors of [GS06] introduced an algebraic curve and an associated polynomial called the  $C$ -polynomial, as well as an operator that shifts  $N$  and the corresponding recursion relation in  $N$  for coloured polynomials implemented by the quantum  $C$ -polynomial. The dependence on  $N$  was subsequently generalized to  $a$ -dependence in [MM21]. Our  $B$ -polynomial is similar to, but different from the  $C$ -polynomial (the definition of the  $B$ -polynomial involves the full coloured HOMFLY-PT polynomial, while that of the  $C$ -polynomial involves coefficients of a cyclotomic expansion), so our claims are accordingly independent of those in [GS06, MM21].

Physically,  $x$  represents an open string modulus that can be identified with a Coulomb branch parameter of a 3d  $\mathcal{N} = 2$  theory associated to a Lagrangian brane and  $a$  is the Kähler parameter of the resolved conifold, which can be identified with a Coulomb branch parameter of a 5d gauge theory. Therefore, the dependence on both  $a$  and  $x$ , as well as their conjugate variables  $b$  and  $y$ , should arise in the description of a coupled 3d-5d system, which means that the  $A$ - and  $B$ -polynomials are not independent but should be viewed as specializations of a higher dimensional variety associated to what we call the  $AB$ -ideal. Likewise, the quantum  $A$ - and  $B$ -polynomials arise as specializations of a  $D$ -module that we call the quantum  $AB$ -ideal, and which quantizes the classical  $AB$ -ideal. This is summarized in the following conjecture.

**Conjecture 2.** *Let us endow  $(\mathbb{C}^*)^4$  with the holomorphic symplectic form*

$$\Omega := d \log x \wedge d \log y + d \log a \wedge d \log b, \quad x, y, a, b \in \mathbb{C}^*.$$

For every knot  $K$ , there is a holomorphic Lagrangian subvariety  $\Gamma_K \subset (\mathbb{C}^*)^4$  with the following properties:

(1) This holomorphic Lagrangian is preserved under the Weyl symmetry

$$x \mapsto a^{-1}x^{-1}, \quad y \mapsto y^{-1}, \quad a \mapsto a, \quad b \mapsto y^{-1}b.$$

(2) The projection of  $\Gamma_K$  on  $(\mathbb{C}^*)_{x,y,a}^3$  is the zero set of the  $a$ -deformed  $A$ -polynomial of  $K$ .

(3) Moreover, if  $\hat{x}, \hat{y}, \hat{a}, \hat{b}$  are operators such that

$$\hat{y}\hat{x} = q\hat{x}\hat{y}, \quad \hat{b}\hat{a} = q\hat{a}\hat{b},$$

and all the other pairs commute, then the ideal defining  $\Gamma_K$  can be quantized to a left ideal  $\hat{\Gamma}_K \subset \mathbb{C}[\hat{x}^{\pm 1}, \hat{y}^{\pm 1}, \hat{a}^{\pm 1}, \hat{b}^{\pm 1}]$  that annihilates  $F_K(x, a, q)$ .

It is natural to extend the above conjecture to the refined case. First, in analogy with (coloured) superpolynomials, we consider a  $t$ -deformation of  $F_K$  [EGG<sup>+</sup>20]. Then we introduce invariants  $F_K^{(\alpha)}(x, a, q, t)$  for various branches. We can observe that in various expressions the variable  $t$  is mixed with  $x$  or  $a$ , again corresponding to a geometric degeneration to a multiple of the unknot conormal, in a way which suggests that there is a natural conjugate variable  $u$  of  $t$ . We conjecture that for a given knot there exists a holomorphic Lagrangian subvariety in  $(\mathbb{C}^*)^6$  that captures the semiclassical properties of the  $t$ -deformation of  $F_K$ . This variety generalizes a Lagrangian variety in  $(\mathbb{C}^*)^4$  from Conjecture 2 and its quantization is a  $D$ -module in variables  $(\hat{x}, \hat{y}, \hat{a}, \hat{b}, \hat{t}, \hat{u})$  that annihilates  $F_K(x, a, q, t)$ . These statements are summarized the following conjecture.

**Conjecture 3.** Let us endow  $(\mathbb{C}^*)^6$  with the holomorphic symplectic form

$$\Omega := d \log x \wedge d \log y + d \log a \wedge d \log b + d \log t \wedge d \log u, \quad x, y, a, b, t, u \in \mathbb{C}^*.$$

For every knot  $K$ , there is a holomorphic Lagrangian subvariety  $\Gamma_K \subset (\mathbb{C}^*)^6$  with the following properties:

(1) This holomorphic Lagrangian is preserved under the Weyl symmetry

$$x \mapsto (-t)^3 a^{-1} x^{-1}, \quad y \mapsto t^s y^{-1}, \quad a \mapsto a, \quad b \mapsto (-t)^{\frac{s}{2}} y^{-1} b, \quad t \mapsto t, \quad u \mapsto x^{-s} y^{-3} a^{-\frac{s}{2}} u,$$

where  $s$  is a version of  $s$ -invariant of the knot  $K$ .

(2) The projection of  $\Gamma_K$  on  $(\mathbb{C}^*)_{x,y,a,t}^4$  is the zero set of the super- $A$ -polynomial of  $K$ .

(3) Moreover, if  $\hat{x}, \hat{y}, \hat{a}, \hat{b}, \hat{t}, \hat{u}$  are operators such that

$$\hat{y}\hat{x} = q\hat{x}\hat{y}, \quad \hat{b}\hat{a} = q\hat{a}\hat{b}, \quad \hat{u}\hat{t} = q\hat{t}\hat{u},$$

and all the other pairs commute, then the ideal defining  $\Gamma_K$  can be quantized to a left ideal  $\hat{\Gamma}_K \subset \mathbb{C}[\hat{x}^{\pm 1}, \hat{y}^{\pm 1}, \hat{a}^{\pm 1}, \hat{b}^{\pm 1}, \hat{t}^{\pm 1}, \hat{u}^{\pm 1}]$  that annihilates  $F_K(x, a, q, t)$ .

We confirm the above conjectures in a number of nontrivial examples throughout the paper.

The rest of the paper is organized as follows. Section 2 summarizes basic information on knot invariants, brane systems,  $\hat{Z}$  and  $F_K$  invariants, and the knots-quivers correspondence. In section 3 we introduce  $F_K$  invariants for various branches of the  $A$ -polynomial and provide various examples of such objects. In section 4 we show that  $F_K$  invariants can be written in the form of quiver generating series, discuss some properties of such series, identify corresponding quivers in various examples, and explain how they can be reconstructed from the  $A$ -polynomial and its branches. Section 5 presents how to reconstruct quivers using  $R$ -matrices or inverted Habiro series. In section 6 we discuss the quantized Coulomb branch from

a more general 3d-5d physical perspective, introduce the  $B$ -polynomial, and show that both the  $A$ - and  $B$ -polynomials are specializations of a higher dimensional complex Lagrangian variety, which has a quantum counterpart that we call the quantum  $AB$ -ideal. Section 7 contains a discussion of other properties of various objects introduced earlier: consistency with constraints imposed by anomalies, subtle properties of vacua of supersymmetric theories corresponding to  $F_K$ , surgeries and relation to closed 3-manifolds, as well as relations to modular tensor categories and logarithmic vertex operator algebras. We conclude with a brief summary of possible future directions of research in section 8.

## 2. PRELIMINARIES

In this section we present background material on knot polynomials, brane configurations and enumerative geometry,  $\widehat{Z}$  and  $F_K$  invariants, as well as the knots-quivers correspondence.

### 2.1. Knot polynomials.

2.1.1. *HOMFLY-PT polynomials.* Let  $K \subset S^3$  be a knot. Its *HOMFLY-PT polynomial*  $P(K; a, q)$  is a topological invariant [HOM<sup>+</sup>85, PT87] which can be calculated via the skein relation:

$$a^{1/2}P(\times) - a^{-1/2}P(\oslash) = (q^{1/2} - q^{-1/2})P(\cup)$$

with a normalisation condition  $P(0_1; a, q) = 1$ . This is called the *reduced* normalisation and corresponds to dividing by the full natural HOMFLY-PT polynomial for the unknot (here denoted by bar):

$$P(K; a, q) = \frac{\bar{P}(K; a, q)}{\bar{P}(0_1; a, q)}, \quad \bar{P}(0_1; a, q) = \frac{a^{1/2} - a^{-1/2}}{q^{1/2} - q^{-1/2}}.$$

For  $a = q^2$ ,  $P(K; a, q)$  reduces to the Jones polynomial  $J(K; q)$  [Jon85], whereas the substitution  $a = q^N$  leads to the  $\mathfrak{sl}_N$  Jones polynomial  $J^{\mathfrak{sl}_N}(K; q)$  [RT90].

More generally, the *coloured HOMFLY-PT polynomials*  $P_R(K; a, q)$  are similar polynomial knot invariants defined as the expectation value of the knot viewed as a Wilson line in the Chern-Simons gauge theory on  $S^3$  [Wit89], which depends also on a representation  $R$  of  $\mathfrak{sl}_N$ . In this setting, the original HOMFLY-PT corresponds to the defining representation. The substitutions  $a = q^2$  and  $a = q^N$  lead to the coloured Jones polynomial  $J_R(K; q)$  and coloured  $\mathfrak{sl}_N$  Jones polynomials  $J_R^{\mathfrak{sl}_N}(K; q)$  respectively. We will be interested mainly in the HOMFLY-PT polynomials coloured by the totally symmetric representations  $R = S^r$  with  $r$  boxes in one row of the Young diagram. In order to simplify the notation, we will denote them by  $P_r(K; a, q)$  and call them simply the HOMFLY-PT polynomials.

There is also a  $t$ -deformation of the HOMFLY-PT polynomials [DGR06, GS12a]. The *superpolynomial*  $\mathcal{P}_r(K, a, q, t)$  is defined as the Poincaré polynomial of the triply-graded homology that categorifies the HOMFLY-PT polynomial:

$$(1) \quad \begin{aligned} P_r(K; a, q) &= \sum_{i,j,k} (-1)^k a^i q^j \dim \mathcal{H}_{i,j,k}^{S^r}(K), \\ \mathcal{P}_r(K; a, q, t) &= \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}_{i,j,k}^{S^r}(K). \end{aligned}$$

From above equations one can immediately see that the superpolynomial reduces to HOMFLY-PT for  $t = -1$ :

$$P_r(K; a, q, -1) = P_r(K; a, q).$$

2.1.2. *A-polynomials.* The *A-polynomial* is a knot invariant coming from the character variety of the complement of a given knot  $K$  in  $S^3$  [CCG<sup>+</sup>94]. It takes the form of an algebraic curve  $A_K(x, y) = 0$ , for  $x, y \in \mathbb{C}^*$ . According to the volume conjecture, it also captures the asymptotics of the coloured Jones polynomials  $J_r(K; q)$  for large colours  $r$ . The quantisation of the *A-polynomial* encodes information about all colours, not only large ones. Namely, it gives the recurrence relations satisfied by  $J_r(K; q)$ , which can be written in the form

$$\hat{A}_K(\hat{x}, \hat{y})J_*(K; q) = 0,$$

where  $\hat{x}$  and  $\hat{y}$  act by

$$\hat{x}J_r = q^r J_r, \quad \hat{y}J_r = J_{r+1},$$

and satisfy the relation  $\hat{y}\hat{x} = q\hat{x}\hat{y}$ . The above conjecture was proposed independently in the context of quantisation of the Chern-Simons theory [Guk05] and in parallel mathematics developments [Gar04]. The operator  $\hat{A}_K(\hat{x}, \hat{y})$  is referred to as the *quantum A-polynomial*; in the classical limit  $q = 1$  it reduces to the *A-polynomial*.

The above conjectures were generalized to coloured HOMFLY-PT polynomials [AV12] and coloured superpolynomials [AGSF12, FGS13], which we briefly introduced in (1). In these cases the objects mentioned in the previous paragraph become  $a$ - and  $t$ -dependent. In particular, the asymptotics of coloured superpolynomials  $\mathcal{P}_r(K; a, q, t)$  for large  $r$  is captured by an algebraic curve called the *super-A-polynomial*, defined by the equation  $A_K(x, y, a, t) = 0$ . For  $t = -1$  it reduces to  $a$ -deformed *A-polynomial*, and upon setting in addition  $a = 1$ , we obtain the original *A-polynomial* as a factor. For brevity, all these objects are often referred to as *A-polynomials*. The quantisation of the super-*A-polynomial* gives rise to quantum super-*A-polynomial*  $\hat{A}_K(\hat{x}, \hat{y}, a, q, t)$ , which is a  $q$ -difference operator that encodes the recurrence relations for the coloured superpolynomials:

$$\hat{A}_K(\hat{x}, \hat{y}, a, q, t)\mathcal{P}_*(K; a, q, t) = 0.$$

A universal framework that enables us to determine a quantum *A-polynomial* from an underlying classical curve  $A(x, y) = 0$  was proposed in [GS12b] (irrespective of extra parameters these curves depend on, and also beyond examples related to knots).

## 2.2. Brane configurations and enumerative geometry.

2.2.1. *Large- $N$  transition.* The physical background of this work can be represented by the system of  $N$  fivebranes supported on  $\mathbb{R}^2 \times S^1 \times Y$ , where  $Y$  is embedded (as the zero-section) inside the Calabi-Yau space  $T^*Y$  and  $\mathbb{R}^2 \times S^1 \subset \mathbb{R}^4 \times S^1$ :

$$(2) \quad \begin{array}{ccc} \text{spacetime :} & \mathbb{R}^4 \times S^1 \times T^*Y & \\ & \cup & \cup \\ N \text{ M5-branes :} & \mathbb{R}^2 \times S^1 \times Y & \end{array}$$

Finding the large- $N$  limit of this system for general  $Y$  is highly nontrivial (see [GPV17, sec.7] and [ES19, Remark 2.4]). However, when  $Y$  is a knot complement  $M_K := S^3 \setminus K$ , there is an equivalent description for which the study of large- $N$  behaviour can be reduced to the celebrated “large- $N$  transition” [GV98, OV00].

We consider first a description without transition. From the viewpoint of 3d-3d correspondence,  $N$  fivebranes on  $Y = M_K$  produce a 4d  $\mathcal{N} = 4$  theory – which is a close cousin of 4d  $\mathcal{N} = 4$  super-Yang-Mills but is *not* 4d  $\mathcal{N} = 4$  SYM – on a half-space  $\mathbb{R}^3 \times \mathbb{R}_+$  coupled to 3d  $\mathcal{N} = 2$  theory  $T[M_K]$  on the boundary. Indeed, near the boundary  $T^2 = \Lambda_K = \partial M_K$ ,

the compactification of  $N$  fivebranes produces a 4d  $\mathcal{N} = 4$  theory which has moduli space of vacua  $\text{Sym}^N(\mathbb{C}^2 \times \mathbb{C}^*)$  [CGPS20]. (The moduli space of vacua in 4d  $\mathcal{N} = 4$  SYM is  $\text{Sym}^N(\mathbb{C}^3)$ .) The  $SU(N)$  gauge symmetry of this theory appears as a global symmetry of the 3d boundary theory  $T[M_K]$ . In particular, the variables  $x_i \in \mathbb{C}^*$  are complexified fugacities for this global (“flavour”) symmetry. For  $G = SU(2)$ , the moduli space of vacua of the knot complement theory  $T[M_K]$  gives precisely the  $A$ -polynomial of  $K$ . Similarly,  $G_{\mathbb{C}}$  character varieties of  $M_K$  are realised as spaces of vacua in  $T[M_K, SU(N)]$  with  $G = SU(N)$  [FGS13, FGSS13].

We next give another equivalent description of the physical system (2) with  $Y = M_K$ , where the large- $N$  behaviour is easier to analyse:

$$(3) \quad \begin{array}{l} \text{spacetime : } \mathbb{R}^4 \times S^1 \times T^*S^3 \\ \qquad \qquad \qquad \cup \qquad \qquad \cup \\ N \text{ M5-branes : } \mathbb{R}^2 \times S^1 \times S^3 \\ \rho \text{ M5'-branes : } \mathbb{R}^2 \times S^1 \times L_K. \end{array}$$

This brane configuration is basically a variant of (2) with  $Y = S^3$  and  $\rho$  extra M5-branes supported on  $\mathbb{R}^2 \times S^1 \times L_K$ , where  $L_K \subset T^*S^3$  is the conormal bundle of the knot  $K \subset S^3$ . There is, however, a crucial difference between fivebranes on  $S^3$  and  $L_K$ . Since the latter are non-compact in two directions orthogonal to  $K$ , they carry no dynamical degrees of freedom away from  $K$ . One can path integrate those degrees of freedom along  $K$ , which effectively removes  $K$  from  $S^3$  and puts the corresponding boundary conditions on the boundary  $T^2 = \partial M_K$ . The resulting system is precisely (2) with  $Y = M_K$ . Equivalently, one can use the topological invariance along  $S^3$  to move the tubular neighbourhood of  $K \subset S^3$  to “infinity”. This creates a long neck isomorphic to  $\mathbb{R} \times T^2$ , as in the above discussion. Either way, we end up with a system of  $N$  fivebranes on the knot complement and no extra branes on  $L_K$ , so that the choice of  $GL(\rho, \mathbb{C})$  flat connection on  $L_K$  is now encoded in the boundary condition for  $SL(N, \mathbb{C})$  connection<sup>1</sup> on  $T^2 = \partial M_K$ . In particular, the latter has at most  $\rho$  nontrivial parameters  $x_i \in \mathbb{C}^*$ ,  $i = 1, \dots, \rho$ .

In this paper we consider the simplest case of  $\rho = 1$ . Then we can use the geometric transition of [GV98], upon which there is one brane on  $L_K$  and  $N$  fivebranes on the zero-section of  $T^*S^3$  disappear. The Calabi-Yau space  $T^*S^3$  undergoes a topology changing transition to a new Calabi-Yau space  $X$ , the so-called “resolved conifold”, which is the total space of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{CP}^1$ , and only the Ooguri-Vafa fivebranes supported on the conormal bundle  $L_K$  remain:

$$(4) \quad \begin{array}{l} \text{spacetime : } \mathbb{R}^4 \times S^1 \times X \\ \qquad \qquad \qquad \cup \qquad \qquad \cup \\ \rho \text{ M5'-branes : } \mathbb{R}^2 \times S^1 \times L_K. \end{array}$$

Note that on the resolved conifold side, i.e. after the geometric transition,  $\log a = \text{Vol}(\mathbb{CP}^1) + i \int B = N\hbar$  is the complexified Kähler parameter which enters the generating function of enumerative invariants.

To summarize, a system of  $N$  fivebranes on a knot complement (2) is equivalent to a brane configuration (4), with a suitable map that relates the boundary conditions in the two cases. There is another system, closely related to (4), that one can obtain from (3) by first

<sup>1</sup>To be more precise, it is a  $GL(N, \mathbb{C})$  connection, but the dynamics of the  $GL(1, \mathbb{C})$  sector is different from that of the  $SL(N, \mathbb{C})$  sector and can be decoupled.

reconnecting  $\rho$  branes on  $L_K$  with  $\rho$  branes on  $S^3$ . This give  $\rho$  branes on  $M_K$  (that go off to infinity just like  $L_K$  does) plus  $N - \rho$  branes on  $S^3$ . Assuming that  $\rho \sim O(1)$  as  $N \rightarrow \infty$  (e.g.  $\rho = 1$  in the context of this paper), after the geometric transition we end up with a system like (4), except  $L_K$  is replaced by  $M_K$  and  $\text{Vol}(\mathbb{C}\mathbb{P}^1) + i \int B = (N - \rho)\hbar$ . Both of these systems on the resolved side compute the HOMFLY-PT polynomials of  $K$  coloured by Young diagrams with at most  $\rho$  rows.

**2.2.2. Twisted superpotential.** The leading, genus-0 contribution to the generating function of enumerative invariants is the twisted superpotential. It can be computed either on the resolved side of the transition, where  $a$  is a Kähler parameter, or on the original (“deformed”) side, for a family of theories labelled by  $N$ . Either way, one finds that the twisted superpotential is given by the double-scaling limit that combines large-colour and semiclassical limits of the HOMFLY-PT polynomials [FGS13, FGSS13]:

$$(5) \quad P_r(K; a, q) \xrightarrow[\hbar \rightarrow 0]{r \rightarrow \infty} \int \prod_i \frac{dz_i}{z_i} \exp \left( \frac{1}{\hbar} \widetilde{\mathcal{W}}_{T[M_K]}(z_i, x, a) + O(\hbar^0) \right),$$

with  $x = q^r$  kept fixed. We can read off the structure of  $T[M_K]$  from the terms in  $\widetilde{\mathcal{W}}_{T[M_K]}(z_i, x, a)$ :

$$(6) \quad \begin{aligned} \text{Li}_2(a^{n_Q} x^{n_M} z_i^{n_{z_i}}) &\longleftrightarrow (\text{chiral field}), \\ \frac{\kappa_{ij}}{2} \log \zeta_i \cdot \log \zeta_j &\longleftrightarrow (\text{Chern-Simons coupling}). \end{aligned}$$

Each dilogarithm is interpreted as the one-loop contribution of a chiral superfield with charges  $(n_Q, n_M, n_{z_i})$  under the global symmetries  $U(1)_Q$  (arising from the internal 2-cycle in  $X$ ) and  $U(1)_M$  (corresponding to the non-dynamical gauge field on the M5-brane), as well as the gauge group  $U(1) \times \dots \times U(1)$ . Quadratic-logarithmic terms are identified with Chern-Simons couplings among the various  $U(1)$  symmetries, with  $\zeta_i$  denoting the respective fugacities.

We can integrate out the dynamical fields (whose VEVs are given by  $\log z_i$ ) using the saddle point approximation to obtain the effective twisted superpotential:

$$(7) \quad \widetilde{\mathcal{W}}_{T[M_K]}^{\text{eff}}(x, a) = \frac{\partial \widetilde{\mathcal{W}}_{T[M_K]}(z_i, x, a)}{\partial \log z_i}.$$

Then after introducing the dual variable  $y$  (the effective Fayet-Iliopoulos parameter), we arrive at the  $A$ -polynomial:

$$(8) \quad \log y = \frac{\partial \widetilde{\mathcal{W}}_{T[M_K]}^{\text{eff}}(x, a)}{\partial \log x} \Leftrightarrow A_K(x, y, a) = 0.$$

**2.2.3. Curve counts.** We recall a geometric picture underlying the quiver description of the invariant  $F_K$ . Recall the curve counting interpretation of  $F_K$  from [EGG+20]. We start from the deformed conifold  $T^*S^3$  and the knot complement  $M_K$ , with the Legendrian conormal  $\Lambda_K \subset ST^*S^3$  as ideal boundary. In case  $K$  is fibered, we shift  $M_K$  off of  $S^3$  along a closed 1-form  $\beta$  that generates  $H^1(M_K) = \mathbb{R}$ . We take this form to agree with the form  $d\mu$  that is dual to the meridian circle on the boundary of a tubular neighbourhood of the knot. If  $K$  is not fibered, such a 1-form necessarily has zeroes and the shift leaves intersection points between  $M_K$  and  $S^3$  that can be normalized to locally appear as cotangent fibers.

We want to count (generalized) holomorphic curves with boundary on  $M_K$ . There are two ways to do this for fibered knots: either we consider  $M_K$  as a Lagrangian submanifold in the resolved conifold  $X$  or we can use a sufficiently SFT-stretched almost complex structure on  $T^*S^3$  for which all curves leave a neighbourhood of the zero section, see [ES19, Section 2.5]. The resulting counts (and in fact the curves) are the same. In the non-fibered case, the second approach still works: after stretching  $M_K$  intersects a neighbourhood of  $S^3$  in  $T^*S^3$  in a finite collection of cotangent fibers. Then possible curves in the inside region (near  $S^3$ ) have boundaries on these fibers and positive punctures at Reeb chords corresponding to geodesics connecting them. The dimension of such a curve is

$$\dim = \sum_j (\text{index}(\gamma_j) + 1) \geq 2,$$

where the sum runs over positive punctures of the curve,  $\gamma_j$  is the Reeb chord at the puncture, and  $\text{index}(\gamma_j)$  is the Morse index of the corresponding geodesic. It follows that no such curve can appear after stretching, since the outside part would then have negative index. This means that there is a curve count also for  $M_K$ . As we will discuss below, although this curve count is well defined and invariant, when intersections between  $S^3$  and  $M_K$  cannot be removed, it is only one point in a space of curve counts that also takes into account certain punctured curves. The present discussion applies to the more involved curve counts of punctured curves as well, although the geometric description is less direct.

In this setting the quiver picture arises from a description of a the Lagrangian  $M_K$  as deformed by a collection of basic holomorphic disks. Here the disks are embedded and deform the standard cotangent neighbourhood of  $M_K$  (where there are no holomorphic curves) to the cotangent bundle with neighbourhoods of these basic disks attached. Such neighbourhoods support holomorphic curves that come from branched covers of the basic disks with constant curves attached and the total count of generalized holomorphic curves is determined by the linking data of the basic disk boundaries. It is given by the quiver partition function, where the nodes correspond to the basic disks and the arrows to the linking information. Here the data of a node is the homological degree of its boundary and the number of wraps around the central  $\mathbb{C}P^1$ . We point out that nodes of boundary degree zero play a special role for non-fibered knots.

### 2.3. $\hat{Z}$ and $F_K$ invariants.

**2.3.1.  $\mathfrak{sl}_2$  and  $\mathfrak{sl}_N$  invariants.** In their study of 3d  $\mathcal{N} = 2$  theories  $T[Y]$  for 3-manifolds  $Y$ , Gukov-Putrov-Vafa [GPV17] and Gukov-Pei-Putrov-Vafa [GPPV20] conjectured the existence of the 3-manifold invariants  $\hat{Z}(Y)$  (also known as “homological blocks” or “GPPV invariants”) valued in  $q$ -series with integer coefficients. These  $q$ -series invariants exhibit peculiar modular properties, the exploration of which was initiated in [BMM20b, CCF<sup>+</sup>19, BMM20a, CFS19].

More recently, Gukov-Manolescu [GM21] introduced a version of  $\hat{Z}$  for knot complements, which they called  $F_K$ . If  $K \subset S^3$  is a knot, then  $F_K = \hat{Z}(S^3 \setminus K)$ . The motivation was to study  $\hat{Z}$  more systematically using Dehn surgery. Recall that the Melvin-Morton-Rozansky expansion [MM95, BNG96, Roz96, Roz98] (also known as “loop expansion” or “large colour expansion”) of the coloured Jones polynomials is the asymptotic expansion near  $\hbar \rightarrow 0$  while

keeping  $x = q^r = e^{r\hbar}$  fixed:

$$J_r(K; q = e^{\hbar}) = \sum_{j \geq 0} \frac{p_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}, \quad p_j(x) \in \mathbb{Z}[x, x^{-1}], \quad p_0 = 1.$$

Here  $\Delta_K(x)$  is the Alexander polynomial of  $K$ . The main conjecture of [GM21] was then the following.

**Conjecture 4.** *For every knot  $K \subset S^3$ , there exists a two-variable series*

$$(9) \quad F_K(x, q) = \frac{1}{2} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} f_m(q)(x^{m/2} - x^{-m/2}), \quad f_m(q) \in \mathbb{Z}[q^{-1}, q]$$

*such that its asymptotic expansion agrees with the Melvin-Morton-Rozansky expansion of the coloured Jones polynomials<sup>2</sup>:*

$$(10) \quad F_K(x, q = e^{\hbar}) = (x^{1/2} - x^{-1/2}) \sum_{j \geq 0} \frac{p_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}.$$

*Moreover, this series is annihilated by the quantum  $A$ -polynomial:*

$$\hat{A}_K(\hat{x}, \hat{y}, q) F_K(x, q) = 0.$$

Let us stress that while the same form of quantum  $A$ -polynomial  $\hat{A}_K(\hat{x}, \hat{y})$  arises in the analysis of coloured Jones polynomial and  $F_K$  invariants, there is a subtle but important difference between these two situations, which has to do with the initial conditions that need to be imposed.

Conjecture 4 concerns  $\mathfrak{g} = \mathfrak{sl}_2$ . An extension to arbitrary  $\mathfrak{g}$  was studied in [Par20a]. In particular, the existence of an  $\mathfrak{sl}_N$  generalisation of  $F_K$ , which we denote by  $F_K^{\mathfrak{sl}_N}$ , was conjectured and it was observed that its specialisation to symmetric representations,  $F_K^{\mathfrak{sl}_N, \text{sym}}$ , is annihilated by the corresponding quantum  $A$ -polynomial:

$$(11) \quad \hat{A}_K(\hat{x}, \hat{y}, a = q^N, q) F_K^{\mathfrak{sl}_N, \text{sym}}(x, q) = 0.$$

2.3.2.  *$a$ -deformed  $F_K$  invariants.* The analysis of [EGG<sup>+</sup>20] showed that  $F_K^{\mathfrak{sl}_N, \text{sym}}(x, q)$  for all  $N$  can be captured by  $a$ -deformed invariants  $F_K(x, a, q)$ :

$$(12) \quad F_K(x, a = q^N, q) = F_K^{\mathfrak{sl}_N, \text{sym}}(x, q).$$

They are annihilated by the quantum  $a$ -deformed  $A$ -polynomials:

$$(13) \quad \hat{A}_K(\hat{x}, \hat{y}, a, q) F_K(x, a, q) = 0,$$

where the operators act as follows:

$$(14) \quad \hat{x} F_K(x, a, q) = x F_K(x, a, q), \quad \hat{y} F_K(x, a, q) = F_K(qx, a, q).$$

Moreover, the asymptotic expansion of  $F_K(x, a, q)$  should agree with that of HOMFLY-PT polynomials. That is

$$(15) \quad \log F_K(e^{r\hbar}, a, e^{\hbar}) = \log P_r(K; a, e^{\hbar})$$

<sup>2</sup>[GM21] uses the unreduced normalisation. In the reduced normalisation, used in the major part of this paper, (10) reads  $F_K(x, q = e^{\hbar}) = \sum_{j \geq 0} \frac{p_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}$ .

as  $\hbar$ -series. In certain cases (including all  $(2, 2p + 1)$  torus knots) lifting this relation to the resummed level leads to a simple substitution:

$$(16) \quad F_K(x, a, q) = P_r(K; a, q)|_{q^r=x},$$

but this fails for many simple knots like  $4_1$ . From the physical point of view,  $F_K^{\text{sl}_N, \text{sym}}(x, q)$  and  $F_K(x, a, q)$  encode BPS spectra on two sides of the large- $N$  transition discussed in Section 2.2.1.

2.3.3. *Normalisation and Conventions.* Before continuing, we make a few remarks on different conventions involving  $F_K$  and  $J_r^{\text{sl}_N}$ .

- In [GM21] and Conjecture 4,  $F_K$  is presented in the *balanced* expansion which involves a summation over both positive and negative powers of  $x$  with Weyl symmetry being manifest. However, for our purposes it will be more natural to work with the *positive* expansion, as in [EGG<sup>+</sup>20]. This means that we express  $F_K$  as a power series in  $x$  expanded around 0. There is also a closely related *negative* expansion coming from expanding around  $x = \infty$  or by applying Weyl symmetry to the positive expansion. The balanced expansion can be rederived by averaging the positive and negative expansions.
- When working with quiver forms (Sections 2.4 and 4), we often treat  $F_K$  as an integer power series starting with 1, see e.g. Equation (23). We stress that this is only correct up to an overall prefactor

$$(17) \quad \exp\left(\frac{p(\log x, \log a)}{\hbar}\right)$$

where  $p$  is a polynomial of degree at most 2. These prefactors are important for some properties of  $F_K$  and can be derived from the  $A$ - and  $B$ -polynomials as in Section 6.5.

- In the literature there are a collection of different normalisations in which  $F_K$  and  $J_r^{\text{sl}_N}$  are presented. The three possibilities correspond to the different values which can be assigned to the unknot invariant.
  - The *reduced* normalisation corresponds to normalizing away the unknot,

$$J_r^{\text{sl}_N}(\mathbf{0}_1, q) = 1 = F_{\mathbf{0}_1}(x, a, q).$$

This is the convention most present in the literature on HOMFLY-PT, superpolynomials, and  $A$ -polynomials, e.g. [DGR06, AGSF12, FGS13, FGSS13, NRZS12, EGG<sup>+</sup>20].

- The *unreduced* normalisation corresponds to normalizing away the denominator of the full unknot factor,

$$J_r^{\text{sl}_N, \text{unreduced}}(\mathbf{0}_1, q) = \frac{(xq; q)_\infty}{(xa; q)_\infty} = F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q).$$

This convention is common in the growing literature on  $F_K$  invariants, e.g. [GM21, Par20a, Par20b, GHN<sup>+</sup>21, EGG<sup>+</sup>20], particularly when studying unknot invariants or working with the balanced expansion of  $F_K$ .

- The *fully unreduced* normalisation corresponds to leaving the full unknot factor intact,

$$J_r^{\text{sl}_N, \text{fully unreduced}}(\mathbf{0}_1, q) = e^{\frac{-\log(x)\log(a)}{2\hbar}} x^{\frac{1}{2}} \frac{(a; q)_\infty (xq; q)_\infty}{(xa; q)_\infty (q; q)_\infty} = F_{\mathbf{0}_1}^{\text{fully unreduced}}(x, a, q).$$

This normalisation is natural in the context of enumerative invariants and can be found in [OV00, AENV14, EN18, EKL20a, EKL20b, ES19, DE20]. In the literature this normalisation is usually called just “unreduced”, but since we join different perspectives, we distinguish it from the one discussed in the previous point.

We primarily work in the reduced normalisation and each use of the other normalisation (e.g. in the analysis of the unknot) is clearly stated.

## 2.4. Knots-quivers correspondence.

2.4.1. *Quivers and their representations.* A quiver  $Q$  is an oriented graph, i.e. a pair  $(Q_0, Q_1)$  where  $Q_0$  is a finite set of vertices and  $Q_1$  is a finite set of arrows between them. We number the vertices by  $1, 2, \dots, m = |Q_0|$ . An adjacency matrix of  $Q$  is the  $m \times m$  integer matrix with entries  $C_{ij}$  equal to the number of arrows from  $i$  to  $j$ . If  $C_{ij} = C_{ji}$ , we call the quiver symmetric.

A quiver representation with a dimension vector  $\mathbf{d} = (d_1, \dots, d_m)$  is an assignment of a vector space of dimension  $d_i$  to the node  $i \in Q_0$  and a linear map  $\gamma_{ij} : \mathbb{C}^{d_i} \rightarrow \mathbb{C}^{d_j}$  to each arrow from vertex  $i$  to vertex  $j$ . Quiver representation theory studies moduli spaces of quiver representations. While explicit expressions for invariants describing those spaces are difficult to find in general, they are quite well understood in the case of symmetric quivers [KS08, KS11, Efi12, MR19, FR18]. Important information about the moduli space of representations of a symmetric quiver is encoded in the *motivic generating series* defined as

$$(18) \quad P_Q(\mathbf{x}, q) = \sum_{\mathbf{d} \geq 0} (-q^{1/2})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{\mathbf{x}^{\mathbf{d}}}{(q; q)_{\mathbf{d}}} = \sum_{d_1, \dots, d_m \geq 0} (-q^{1/2})^{\sum_{i,j} C_{ij} d_i d_j} \prod_{i=1}^m \frac{x_i^{d_i}}{(q; q)_{d_i}},$$

where the denominator is the  $q$ -Pochhammer symbol:

$$(z; q)_n = \prod_{k=0}^{n-1} (1 - zq^k).$$

When  $z = q$ , we will simplify the notation and write  $(q)_n$  and  $(q)_{\mathbf{d}}$  instead of  $(q; q)_n$ , and  $(q; q)_{\mathbf{d}} = \prod_i (q; q)_i$  respectively.

Let us define the *plethystic exponential* of  $f = \sum_n a_n t^n$ ,  $a_0 = 0$  in the following way:

$$\text{Exp}(f)(t) = \exp \left( \sum_k \frac{1}{k} f(t^k) \right) = \prod_n (1 - t^n)^{a_n}.$$

Then we can write

$$(19) \quad P_Q(\mathbf{x}, q) = \text{Exp} \left( \frac{\Omega(\mathbf{x}, q)}{1 - q} \right),$$

$$\Omega(\mathbf{x}, q) = \sum_{\mathbf{d}, s} \Omega_{\mathbf{d}, s} \mathbf{x}^{\mathbf{d}} q^{s/2} = \sum_{\mathbf{d}, s} \Omega_{(d_1, \dots, d_m), s} \left( \prod_i x_i^{d_i} \right) q^{s/2},$$

where  $\Omega_{\mathbf{d}, s}$  are motivic Donaldson-Thomas (DT) invariants [KS08, KS11]. The DT invariants have two geometric interpretations, either as the intersection homology Betti numbers of the moduli space of all semi-simple representations of  $Q$  of dimension vector  $\mathbf{d}$ , or as the Chow-Betti numbers of the moduli space of all simple representations of  $Q$  of dimension vector  $\mathbf{d}$ ;

see [MR19, FR18]. [Efi12] provides a proof of integrality of DT invariants for the symmetric quivers.

2.4.2. *Knots-quivers correspondence for knot conormals.* In the context of the knots-quivers correspondence, we combine  $P_r(K; a, q)$  into the HOMFLY-PT generating series:

$$P_K(y, a, q) = \sum_{r=0}^{\infty} \frac{y^{-r}}{(q)_r} P_r(K; a, q).$$

Using this expression we can encode the Labastida-Mariño-Ooguri-Vafa (LMOV) invariants [OV00, LM01, LMV00] in the following way:

$$(20) \quad P_K(y, a, q) = \text{Exp} \left( \frac{N(y, a, q)}{1 - q} \right), \quad N(y, a, q) = \sum_{r,i,j} N_{r,i,j} y^{-r} a^{i/2} q^{j/2}.$$

According to the LMOV conjecture [OV00, LM01, LMV00],  $N_{r,i,j}$  are integer numbers counting BPS states in the effective 3d  $\mathcal{N} = 2$  theories described in Section 2.2.

The knots-quivers correspondence for the knot conormals [KRSS17, KRSS19] is an assignment of a symmetric quiver  $Q$  (with adjacency matrix  $C$ ), vector  $\mathbf{n} = (n_1, \dots, n_m)$  with integer entries, and vectors  $\mathbf{a} = (a_1, \dots, a_m)$ ,  $\mathbf{l} = (l_1, \dots, l_m)$  with half-integer entries to a given knot  $K$  in such a way that

$$(21) \quad P_K(y, a, q) = \sum_{\mathbf{d} \geq 0} (-q^{1/2})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{y^{\mathbf{n} \cdot \mathbf{d}} a^{\mathbf{a} \cdot \mathbf{d}} q^{\mathbf{l} \cdot \mathbf{d}}}{(q)_{\mathbf{d}}} = P_Q(\mathbf{x}, q)|_{x_i = y^{n_i} a^{a_i} q^{l_i}}.$$

The possibility of such assignment was proven for all 2-bridge knots in [SW19] and for all arborescent knots in [SW21]. Some exotic cases with  $n_i < -1$  (the simplest examples are  $\mathbf{9}_{42}$  and  $\mathbf{10}_{132}$ ) require a generalisation of the correspondence, for more details see [EKL21].

Equation (21) can be rewritten as

$$(22) \quad N(y, a, q) = \Omega(\mathbf{x}, q)|_{x_i = y^{n_i} a^{a_i} q^{l_i}},$$

which ties the knots-quivers correspondence with LMOV conjecture using the fact that DT invariants of symmetric quivers are integer.

2.4.3. *Knots-quivers correspondence for knot complements.* The knots-quivers correspondence can be generalized to knot complements [Kuc20]. Then it is an assignment of a symmetric quiver  $Q$ , an integer  $n_i$ , and half-integers,  $a_i, l_i, i \in Q_0$  to a given knot complement  $M_K = S^3 \setminus K$  in such a way that

$$(23) \quad F_K(x, a, q) = \sum_{\mathbf{d} \geq 0} (-q^{1/2})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} a^{\mathbf{a} \cdot \mathbf{d}} q^{\mathbf{l} \cdot \mathbf{d}}}{(q)_{\mathbf{d}}} = P_Q(\mathbf{x}, q)|_{x_i = x^{n_i} a^{a_i} q^{l_i}}.$$

Due to the limitations of available data, the possibility of such assignment was proven so far only for  $(2, 2p + 1)$  torus knots. Sections 4 and 5 provide evidence for (23) for many new cases of knot complements (however, exotic cases like  $\mathbf{9}_{42}$  and  $\mathbf{10}_{132}$  are still out of range).

Using (19), one can define analogues of LMOV invariants for knot complements [Kuc20]:

$$N(x, a, q) = \Omega(\mathbf{x}, q)|_{x_i = x^{n_i} a^{a_i} q^{l_i}},$$

which can be encoded in the  $F_K$  invariants in analogy to (20):

$$(24) \quad F_K(x, a, q) = \text{Exp} \left( \frac{N(x, a, q)}{1 - q} \right).$$

The integrality of DT, LMOV, and analogous invariants has a natural interpretation in terms of counting BPS states in effective 3d  $\mathcal{N} = 2$  theories.  $N(x, a, q)$  and  $\Omega(\mathbf{x}, q)$  count BPS states in theories  $T[M_K]$  and  $T[Q]$  respectively. Note that basing on Section 2.3, we can rewrite (5) using  $F_K$  invariant:

$$(25) \quad F_K(x, a, q) \xrightarrow{h \rightarrow 0} \int \prod_i \frac{dz_i}{z_i} \exp \left( \frac{1}{h} \widetilde{\mathcal{W}}_{T[M_K]}(z_i, x, a) + \mathcal{O}(h^0) \right).$$

Similarly, the structure of  $T[Q]$  is encoded in the semiclassical limit of the motivic generating series [EKL20b]:

$$(26) \quad P_Q(\mathbf{x}, q) \xrightarrow{q^{d_i=y_i} \atop h \rightarrow 0} \int \prod_i \frac{dy_i}{y_i} \exp \left[ \frac{1}{h} \widetilde{\mathcal{W}}_{T[Q]}(\mathbf{x}, \mathbf{y}) + \mathcal{O}(h^0) \right],$$

$$\widetilde{\mathcal{W}}_{T[Q]}(\mathbf{x}, \mathbf{y}) = \sum_i \text{Li}_2(y_i) + \log \left( (-1)^{C_{ii}} x_i \right) \log y_i + \sum_{i,j} \frac{C_{ij}}{2} \log y_i \log y_j.$$

Using the dictionary (6), we can interpret the elements of (26) in the following way:

- The integral  $\int \prod_i \frac{dy_i}{y_i}$  corresponds to having the gauge group  $U(1)^{(1)} \times \cdots \times U(1)^{(m)}$ ,
- $\text{Li}_2(y_i)$  represents the chiral field with charge 1 under  $U(1)^{(i)}$ ,
- $\frac{C_{ij}}{2} \log y_i \log y_j$  corresponds to the gauge Chern-Simons couplings,  $\kappa_{ij}^{\text{eff}} = C_{ij}$ ,
- $\log \left( (-1)^{C_{ii}} x_i \right) \log y_i$  represents the Chern-Simons coupling between a gauge symmetry and its dual topological symmetry (the Fayet-Iliopoulos coupling).

The saddle point of the twisted superpotential encodes the moduli space of vacua of  $T[Q]$  and defines the quiver  $A$ -polynomials [EKL20b, EKL20a, PSS18, PS19]:

$$(27) \quad \frac{\partial \widetilde{\mathcal{W}}_{T[Q]}(\mathbf{x}, \mathbf{y})}{\partial \log y_i} = 0 \quad \Leftrightarrow \quad A_i(\mathbf{x}, \mathbf{y}) = 1 - y_i - x_i (-y_i)^{C_{ii}} \prod_{j \neq i} y_j^{C_{ij}} = 0.$$

$A_i(\mathbf{x}, \mathbf{y})$  is a classical limit of the quantum quiver  $A$ -polynomial, which annihilates the motivic generating series:

$$\begin{aligned} \hat{A}_i(\hat{\mathbf{x}}, \hat{\mathbf{y}}, q) P_Q(\mathbf{x}, q) &= 0, \\ \hat{x}_i P_Q(x_1, \dots, x_i, \dots, x_m, q) &= x_i P_Q(x_1, \dots, x_i, \dots, x_m, q), \\ \hat{y}_i P_Q(x_1, \dots, x_i, \dots, x_m, q) &= P_Q(x_1, \dots, q x_i, \dots, x_m, q). \end{aligned}$$

The general formula for the quantum quiver  $A$ -polynomial corresponding to the quiver with adjacency matrix  $C$  is given by [EKL20a]

$$(28) \quad \hat{A}_i(\hat{\mathbf{x}}, \hat{\mathbf{y}}, q) = 1 - \hat{y}_i - \hat{x}_i (-q^{1/2} \hat{y}_i)^{C_{ii}} \prod_{j \neq i} \hat{y}_j^{C_{ij}},$$

and we can see that

$$A_i(\mathbf{x}, \mathbf{y}) = \lim_{q \rightarrow 1} \hat{A}_i(\hat{\mathbf{x}}, \hat{\mathbf{y}}, q).$$

2.4.4. *Quiver equivalences.* Since the formulation of the knots-quivers correspondence it has been clear that it is not a bijection and more than one symmetric quiver can correspond to the same knot [KRSS19] – such quivers are called *equivalent*. Later, the study of geometric and physical interpretations [EKL20a, EKL20b] lead to the formulation of quiver transformations

that preserve the motivic generating function (18): unlinking, linking, and removing (or adding) a redundant pair of nodes.

- The unlinking of nodes  $a, b \in Q_0$  is defined as a transformation of  $Q$  leading to a new quiver  $Q'$  such that there is a new node  $n$ :  $Q'_0 = Q_0 \cup n$ ,  $x'_n = q^{-1}x_ax_b$  ( $x'_i = x_i$  for all  $i \in Q_0$ ), and the number of arrows of the new quiver is given by

$$\begin{aligned} C'_{ab} &= C_{ab} - 1, & C'_{nn} &= C_{aa} + 2C_{ab} + C_{bb} - 1, \\ C'_{in} &= C_{ai} + C_{bi} - \delta_{ai} - \delta_{bi}, & C'_{ij} &= C_{ij} \quad \text{for all other cases,} \end{aligned}$$

where  $\delta_{ij}$  is the Kronecker delta.

- The linking of nodes  $a, b \in Q_0$  is defined as a transformation of  $Q$  leading to a new quiver  $Q'$  such that there is a new node  $n$ :  $Q'_0 = Q_0 \cup n$ ,  $x'_n = x_ax_b$  ( $x'_i = x_i$  for all  $i \in Q_0$ ), and the number of arrows of the new quiver is given by

$$\begin{aligned} C'_{ab} &= C_{ab} + 1, & C'_{nn} &= C_{aa} + 2C_{ab} + C_{bb}, \\ C'_{in} &= C_{ai} + C_{bi}, & C'_{ij} &= C_{ij} \quad \text{for all other cases.} \end{aligned}$$

- The pair of nodes  $a, b \in Q_0$  is redundant if  $x_a = qx_b$ ,  $C_{aa} = C_{ab} = C_{bb} - 1$ , and  $C_{ai} = C_{bi}$  for all  $i \in Q_0 \setminus \{a, b\}$ .

A general analysis of equivalent quivers has been conducted recently in [JKL+21]. In this work the unlinking was used to study equivalences of symmetric quivers  $Q$  and  $Q'$  such that  $Q'_0 = Q_0$  and  $x'_i = x_i$  for all  $i \in Q_0$ ; in consequence it was proved that if  $Q$  and  $Q'$  are related by a sequence of disjoint transpositions, each exchanging non-diagonal elements

$$C_{ab} \leftrightarrow C_{cd}, \quad C_{ba} \leftrightarrow C_{dc},$$

for some pairwise different  $a, b, c, d \in Q_0$ , such that

$$\lambda_a \lambda_b = \lambda_c \lambda_d$$

and

$$C_{ab} = C_{cd} - 1, \quad C_{ai} + C_{bi} = C_{ci} + C_{di} - \delta_{ci} - \delta_{di}, \quad \forall i \in Q_0,$$

or

$$C_{cd} = C_{ab} - 1, \quad C_{ci} + C_{di} = C_{ai} + C_{bi} - \delta_{ai} - \delta_{bi}, \quad \forall i \in Q_0,$$

then  $Q$  and  $Q'$  are equivalent. Moreover, these conditions were conjectured to be necessary for  $Q$  and  $Q'$  to correspond to the same knot.

It turns out that transformations presented above can be successfully applied to quivers corresponding to knot complements, which will be useful in Section 4.

### 3. $F_K$ FOR VARIOUS BRANCHES

The perturbative invariants of complex Chern-Simons theory are extensively studied in [DGLZ09]. While the method of that paper allowed the computation of perturbative invariants up to any order, for many years they were not widely used. Recently, in [GM21], it was found that they can be nicely packaged into a two-variable series associated to each knot complement. More precisely, the authors of [GM21] conjecture that the *abelian branch*<sup>3</sup> perturbative invariants can be resummed nicely into a power series in  $q$  and  $x$  with integer coefficients:

$$F_K^{(\text{ab})}(x, q) \stackrel{q=e^h}{=} e^{\frac{1}{h}(S_0^{(\text{ab})}(x) + S_1^{(\text{ab})}(x)h + S_2^{(\text{ab})}(x)h^2 + \dots)}.$$

<sup>3</sup>At finite  $N$ , abelian branch denotes the solution  $y^{(\text{ab})} = 1$  of the classical  $A$ -polynomial.

The relation between the perturbative invariants  $S_j^{(\text{ab})}(x)$  and the two-variable series  $F_K^{(\text{ab})}(x, q)$  can be thought of as the relation between the Gromov-Witten invariants and the Donaldson-Thomas (or BPS) invariants, and in this sense  $F_K(x, q)$  is the non-perturbative complex Chern-Simons partition function associated to the abelian branch. See also [EGG<sup>+</sup>20] for an account of this story from the point of view of topological strings. In order to simplify notation, we will drop the superscript <sup>(ab)</sup> and understand that  $F_K(x, q)$  corresponds to the abelian branch.

It should be noted that the method of [DGLZ09] applies to all branches, so we have perturbative invariants associated to each branch  $y^{(\alpha)}(x)$ . This immediately brings the following question.

**Question 1.** Are there  $F_K^{(\alpha)}(x, q)$  for other branches  $y^{(\alpha)}(x)$ ? That is, can we resum  $e^{\frac{1}{\hbar}(S_0^{(\alpha)}(x)+S_1^{(\alpha)}(x)\hbar+S_2^{(\alpha)}(x)\hbar^2+\dots)}$  into a two-variable series with integrality?

Intriguingly, through various examples we find that the answer to this question seems to be positive. There are indeed  $F_K$  for other branches that can serve as the non-perturbative Chern-Simons partition functions for the knot complements.<sup>4</sup> To explain how to get the series  $F_K^{(\alpha)}(x, q)$  for various branches, in the following subsection we take a closer look at the Newton polygon of the  $A$ -polynomial.

**3.1. The edges and branches of the  $A$ -polynomial.** The Newton polygon of the  $A$ -polynomial contains a wealth of information about the knot. For instance, as discussed in [CCG<sup>+</sup>94], the slope of each edge of the Newton polygon equals the boundary slope of an incompressible surface of the knot complement. For our purposes, the important aspect is a correspondence between the edges of the Newton polygon and the branches of the  $A$ -polynomial, which we explain below.

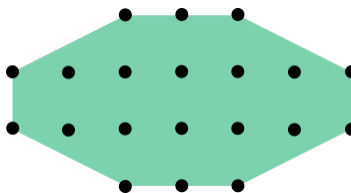


FIGURE 1. The Newton polygon for  $A_4$

Solving the classical  $A$ -polynomial equation  $A_K(x, y) = 0$  for  $y$ , we get different branches  $y^{(\alpha)}(x)$ . We are interested in the behaviour of  $y^{(\alpha)}(x)$  near  $x = 0$  (or  $x = \infty$ , via Weyl symmetry). Consider the Newton polygon of  $A_K$ . As an example, the Newton polygon for the figure-eight knot is depicted in Figure 1. The horizontal direction represents the  $x$ -degree and the vertical direction represents the  $y$ -degree. When  $x$  is close to 0, the dominant terms are the vertices of the Newton polygon that are left-most among the vertices on the same horizontal line. Let us call such vertices “left-vertices”. The equation  $A_K(x, y^{(\alpha)}(x)) = 0$  requires that near  $x = 0$  we should asymptotically have  $y^{(\alpha)}(x) \sim x^{-\frac{n_x}{n_y}}$  for some slope  $\frac{n_y}{n_x}$  of an edge spanned by two of the left-vertices. Let us call such an edge a “left-edge”. Moreover, for any given slope of a left-edge, we can construct a classical solution  $y^{(\alpha)}(x)$

<sup>4</sup>Physically,  $F_K^{(\alpha)}(x, q)$  is a “half-index” of a 2d/3d combined system [GGP14] for the 3d theory  $T[S^3 \setminus K]$  with a 2d  $(0, 2)$  boundary condition labeled by  $\alpha$  and a discrete flux, whose fugacity is  $x$ , cf. [GPV17, GM21].

using the asymptotic dictated by the slope. Therefore, we have just shown the following proposition.

**Proposition 1.** *For any choice of branch  $\alpha$ , there is a left-edge  $e_\alpha$  of the Newton polygon with slope  $\frac{n_y}{n_x}$ , such that*

$$(29) \quad \lim_{x \rightarrow 0} y^{(\alpha)}(x) x^{\frac{n_x}{n_y}} = C$$

for some non-zero constant  $C$ . Moreover, the map  $E : \alpha \mapsto e_\alpha$  is a surjective map onto the set of left-edges.

While the number of branches  $y^{(\alpha)}$  is the same as the total  $y$ -degree (which is the height of the Newton polygon), the number of left-edges is at most the height of the Newton polygon. Therefore,  $E$  is not injective in general.

**Proposition 2.** *For any left-edge  $e$ , the number of pre-images of  $E$  is  $n_y$ , the  $y$ -height of the edge  $e$ .*

Take any left-edge  $e$  with  $x$ -width  $n_x$  and  $y$ -height  $n_y$ . Our convention is such that  $n_y > 0$  but  $n_x$  can be negative. When  $n_y = 1$ , it is easy to see that there is a unique solution with the initial condition (29). When  $n_y > 0$  but  $n_x$  is coprime with  $n_y$  so that the edge is non-degenerate (the endpoints are the only vertices on this edge), then (29) represents  $n_y$  different initial conditions which differ by multiplication by an  $n_y$ -th root of unity. Each of these initial conditions gives a unique classical solution, and therefore there are  $n_y$  number of branches associated to the edge. When  $n_x$  is not coprime with  $n_y$ , the edge is degenerate (can be broken into smaller edges), and there are some genuine multiplicities associated to the initial condition (29). We will see them in an example of the edge of slope  $\infty$  for the knot  $5_2$ .

Our discussion so far can be summarized in the following diagram:

$$\boxed{\text{Left-edge } e \text{ of the Newton polygon} \leftrightarrow n_y \text{ number of branches}}$$

**Remark 1.** If we look at the behaviour of  $y^{(\alpha)}(x)$  near  $x = \infty$ , their asymptotics are determined by the slopes of the right-edge of the Newton polygon. Due to Weyl symmetry, the Newton polygon is symmetric under half-rotation, so we basically get the same set of information.

**Remark 2.** We can similarly solve  $A_K(x, y) = 0$  for  $x$ . With the role of  $x$  and  $y$  switched, everything we described in this section holds.

**Remark 3.** In terms of tropical geometry, what we delineated above can be described using Gröbner fans (See e.g. [Stu96]). A choice of an edge of the  $A$ -polynomial corresponds to a choice of an equivalence class of weight vectors that induce the same initial ideal. When the ideal is neither principal nor homogeneous, there may be no polytope whose normal fan is the Gröbner fan. Therefore, in such cases it is better to think in terms of the ideal and its Gröbner fans.

**Remark 4.** When the  $y$ -height  $n_y$  of  $e_\alpha$  is 1,  $y^{(\alpha)}(x)$  is a power series in  $x$ . In general, however, when  $n_y > 1$ ,  $y^{(\alpha)}(x)$  is a Puiseux series in  $x$ .

3.2.  **$F_K$  from the edges.** In the previous section, we have seen the correspondence between the set of classical solutions and the edges of the Newton polygon. In this section, we will study the quantum version of this correspondence. We are interested in solving the  $q$ -difference equation

$$\hat{A}_K F_K = 0,$$

where  $F_K$  is expanded near  $x = 0$ .<sup>5</sup> For each left-edge, there are some natural initial conditions for the recursion that we can put, which in the semiclassical limit become the classical initial conditions that we studied in the previous section.

**Conjecture 5.** *For each left-edge  $e$  with slope  $\frac{n_y}{n_x}$ , there is a solution to the  $q$ -difference equation  $\hat{A}_K F_K(x, q)$  of the form*

$$F_K^{(\alpha)}(x, q) = e^{\frac{1}{\hbar}(-\frac{1}{2}\frac{n_x}{n_y}(\log x)^2 + \log C \log x)} \left(1 + \sum_{j \geq 1} f_j(q) x^{\frac{j}{d}}\right),$$

where  $d = \frac{n_y}{m}$  in case  $e$  is broken into  $m$  non-degenerate edges,  $C$  is a monomial in  $q$  determined by the coefficients of the vertices of  $e$ , and  $f_j(q)$  are rational functions in  $q$ , which can be expanded into  $q$ -series with integer coefficients.

**Remark 5.** When  $n_y = 1$ , this conjecture is a theorem. This is because we can recursively solve for  $F_K^{(\alpha)}$  in a unique way, analogously to the way it was done for the abelian branch in [EGG<sup>+</sup>20].

**Remark 6.** In the  $a$ -deformed setting, the exponential prefactor will be of the form

$$\exp\left(\frac{p(\log x, \log a)}{\hbar}\right),$$

where  $p$  is a polynomial of degree at most 2.

Along the arguments of the previous section, we see that

- if  $n_y = 1$ , then there is a unique such solution,
- if the edge is non-degenerate but  $n_y > 1$ , then all the  $n_y$  solutions are uniquely determined,
- and if the edge is degenerate, then there are multiple solutions (the number of solutions is the same as the number of branches associated to the edge).

In this way, the solutions to the set of initial conditions determined by the left-edges span the whole  $\deg_y A_K$ -dimensional space of wave functions. We claim that these solutions are exactly the  $F_K^{(\alpha)}$  for various branches  $\alpha$  we mentioned in the beginning of this section (possibly up to an overall factor that is independent of  $x$ ). We can formulate this in the form of the following conjecture.

**Conjecture 6.** *Given a knot  $K$ , for every branch  $y^{(\alpha)}(x)$  of the  $A$ -polynomial, there is a function  $F_K^{(\alpha)}(x, q)$  that is the non-perturbative partition function of the complex Chern-Simons theory in the following sense:*

- (1)  $\hat{A}_K F_K^{(\alpha)} = 0$  with the initial conditions as in Conjecture 5.

---

<sup>5</sup>By the two remarks in the previous subsection, we can do the same for the expansion near  $x = \infty$  or with  $y$  instead of  $x$ .

(2) It is associated to the branch  $y^{(\alpha)}$  in the sense that

$$\lim_{q \rightarrow 1} \frac{F_K^{(\alpha)}(qx, q)}{F_K^{(\alpha)}(x, q)} = y^{(\alpha)}(x).$$

(3) It agrees with the perturbative invariant of [DGLZ09] if we set  $q = e^h$ .

**Remark 7.** There is always an abelian branch associated to the vertical edge whose corresponding initial condition gives the usual  $F_K^{(\text{ab})} = F_K$ , as studied in [GM21, EGG<sup>+</sup>20].

**Remark 8.** It is straightforward to generalize everything we discussed in this section to  $\mathfrak{sl}_N$  and the  $a$ -deformed setup. As we will see later in Section 6, we can even consider the branches of  $b$ , a variable that is the conjugate of  $a$ . In that context, we consider solutions to  $q$ -difference equations with respect to the variable  $a$ . We will see that the branches of  $b$  are canonically in one-to-one correspondence with the branches of  $y$ .

### 3.3. Examples.

3.3.1. *Trefoil.* The quantum  $A$ -polynomial for the right-handed trefoil knot is given by

$$\hat{A}_{\mathfrak{3}_1^r}(\hat{x}, \hat{y}, a, q) = a_0^{\mathfrak{3}_1^r}(\hat{x}, a, q) + a_1^{\mathfrak{3}_1^r}(\hat{x}, a, q)\hat{y} + a_2^{\mathfrak{3}_1^r}(\hat{x}, a, q)\hat{y}^2,$$

with

$$\begin{aligned} a_0^{\mathfrak{3}_1^r}(x, a, q) &= -aq(1-x)(1-qax^2), \\ a_1^{\mathfrak{3}_1^r}(x, a, q) &= (1-ax^2)(a^2x^2 - q^3ax^2 - qax(1+x-ax(1-x)) + q^2(1+a^2x^4)), \\ a_2^{\mathfrak{3}_1^r}(x, a, q) &= qa^2x^3(1-ax)(q-ax^2). \end{aligned}$$

In the classical limit, after modding out by the factor  $(1-ax^2)$ , the Newton polygon is illustrated in Figure 2. While the abelian branch  $F_{\mathfrak{3}_1^r}$  is discussed in detail in [EGG<sup>+</sup>20], let

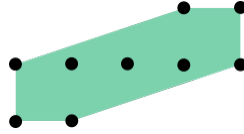


FIGURE 2. The Newton polygon of  $A_{\mathfrak{3}_1^r}$

us briefly review how to compute it. First, observe that near  $x = 0$

$$a_0^{\mathfrak{3}_1^r}(x, a, q) = -qa + O(x^1), \quad (q^{-1}a)a_1^{\mathfrak{3}_1^r}(x, a, q) = qa + O(x^1), \quad a_2^{\mathfrak{3}_1^r}(x, a, q) = O(x^3).$$

The first two non-vanishing  $O(x^0)$  terms correspond exactly to the vertical left-edge of the Newton polygon. Note that we multiplied the coefficients by powers of  $q^{-1}a$  to make the sum of the two  $O(x^0)$  terms vanish. This means our initial condition for solving the recursion is such that

$$F_{\mathfrak{3}_1^r}(x, a, q) = e^{\frac{\log x \log a}{\log q}} x^{-1} (1 + O(x^1)).$$

We can recursively solve the subsequent terms and get

$$F_{\mathfrak{3}_1^r}(x, a, q) = e^{\frac{\log x \log a}{\log q}} x^{-1} \left( 1 + \frac{q-a}{1-q}x + \frac{q^2 + (-2q - q^2 + q^3)a + (1+q-q^2)a^2}{(1-q)(1-q^2)}x^2 + \dots \right),$$

up to an overall factor independent of  $x$ .

For the non-abelian branch of slope  $\frac{1}{3}$ , we need to consider the coefficients of the quantum  $A$ -polynomial near  $y^{-1}x^3 = 0$ . After multiplying appropriate factors, we can make the sum of the terms on this left-edge vanish:

$$\begin{aligned} a_0^{\mathbf{3}_1^r}(x, a, q) &= O(x^0), \\ q^{-\frac{3}{2}1^2}(-q^{\frac{9}{2}}a^{-2}x^{-3})a_1^{\mathbf{3}_1^r}(x, a, q) &= -q^5a^{-2}x^{-3} + O(x^{-2}), \\ q^{-\frac{3}{2}2^2}(-q^{\frac{9}{2}}a^{-2}x^{-3})^2a_2^{\mathbf{3}_1^r}(x, a, q) &= q^5a^{-2}x^{-3} + O(x^{-2}). \end{aligned}$$

The extra factors we had to multiply by mean that the initial condition for solving the recursion is such that

$$F_{\mathbf{3}_1^r}^{(\frac{1}{3})}(x, a, q) = e^{\frac{-\frac{3}{2}(\log x)^2 + \log x \log(-a^{-2})}{\log q}} x^{\frac{9}{2}} (1 + O(x^1)).$$

We can recursively solve the subsequent terms and get

$$F_{\mathbf{3}_1^r}^{(\frac{1}{3})}(x, a, q) = e^{\frac{-\frac{3}{2}(\log x)^2 + \log x \log(-a^{-2})}{\log q}} x^{\frac{9}{2}} \left( 1 + \frac{a}{q}x + \frac{a(q-a)}{q(1-q)}x^2 + \dots \right),$$

up to an overall factor independent of  $x$ .

**3.3.2. Figure-eight.** The Newton polygon for the  $A$ -polynomial of the figure-eight knot is shown in Figure 1. Since all the left-edges are non-degenerate of height 1, everything can be solved term by term in a unique way, just like for the trefoil knot. Following the same procedure, we get the following series for the abelian branch [EGG<sup>+</sup>20]:

$$\begin{aligned} F_{\mathbf{4}_1}(x, a, q) &= e^{\frac{\log x \log a}{\log q}} x^{-1} \left( 1 + \frac{3(q-a)}{1-q}x \right. \\ &\quad \left. + \frac{(q+6q^2+2q^3) - (1+8q+8q^2+q^3)a + (2+6q+q^2)a^2}{(1-q)(1-q^2)}x^2 + \dots \right), \end{aligned}$$

up to an overall factor independent of  $x$ .

For the non-abelian branch of slope  $-\frac{1}{2}$ , we get

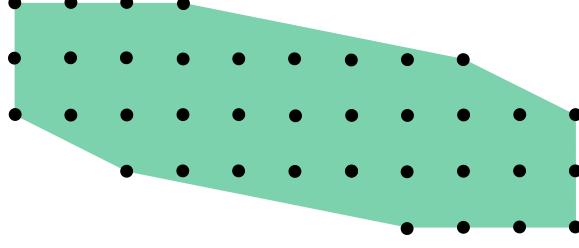
$$\begin{aligned} F_{\mathbf{4}_1}^{(-\frac{1}{2})}(x, a, q) &= e^{\frac{(\log x)^2 + \log x \log a}{\log q}} x^{-1} \left( 1 + \frac{q-2q^2}{1-q}x \right. \\ &\quad \left. + \frac{(q^2-2q^3-2q^4+3q^5+q^6) - q(1-q)(1-q^2)a}{(1-q)(1-q^2)}x^2 + \dots \right), \end{aligned}$$

up to an overall factor independent of  $x$ .

The other non-abelian branch (with slope  $\frac{1}{2}$ ) is conjugate to this one, and the corresponding  $F_K$  can be obtained easily from this one by inverting  $q$ ,  $a$  and  $x$  and then using the Weyl symmetry of [EGG<sup>+</sup>20].

**3.3.3.  $\mathbf{5}_2$  knot.** The Newton polygon of  $A_{\mathbf{5}_2}$  is given in Figure 3. We see that there are 4 branches in total. Two of them have slope  $\infty$ , one has slope  $-\frac{1}{2}$ , and the last one has slope  $-\frac{1}{5}$ . The branches of slope  $-\frac{1}{2}$  and  $-\frac{1}{5}$  are non-degenerate with height 1, and it is straightforward to compute the corresponding  $F_K$  invariants by solving the quantum  $A$ -polynomial recursion. We will focus on the branches of slope  $\infty$ .

One of the branches of slope  $\infty$  is the abelian one, the other is non-abelian. Let us call them (ab) and ( $\infty$ ) respectively. Here we explain how to compute  $F_{\mathbf{5}_2}^{(\alpha)}$  for the two branches,

FIGURE 3. The Newton polygon of  $A_{\mathfrak{5}_2}$ 

with  $\mathfrak{sl}_N$  gauge algebra, using *inverted Habiro series* of [Par21]. In the next section we will provide an alternative way to obtain  $F_{\mathfrak{5}_2}^{(\text{ab})}(x, a, q)$  from the quiver.

About 20 years ago, Habiro showed [Hab02, Hab07] that the coloured Jones polynomials can be decomposed in the following way.

$$J_n(K; q) = \sum_{m \geq 0} a_m(K; q) \prod_{1 \leq j \leq m} (x + x^{-1} - q^j - q^{-j}) \Big|_{x=q^n}.$$

Here  $J_n(K; q)$  denotes the coloured Jones polynomial of  $K$  coloured by the  $n$ -dimensional irreducible representation of  $\mathfrak{sl}_2$ , and  $\{a_m(K; q)\}_{m \geq 0}$  is a sequence of Laurent polynomials in  $q$  with integer coefficients. The analogue of Habiro series for coloured HOMFLY-PT polynomials  $P_r(K; a, q)$  coloured by symmetric representations was studied in [IMMM12, MMM13], and it is given by:

$$P_r(K; a, q) = \sum_{m \geq 0} a_m(K; a, q) \begin{bmatrix} r \\ m \end{bmatrix} \prod_{1 \leq j \leq m} (a^{\frac{1}{2}} q^{\frac{r+j-1}{2}} - a^{-\frac{1}{2}} q^{-\frac{r+j-1}{2}})(a^{\frac{1}{2}} q^{\frac{j-2}{2}} - a^{-\frac{1}{2}} q^{-\frac{j-2}{2}}).$$

The sequences of cyclotomic coefficients  $\{a_m(K; q)\}$  and  $\{a_m(K; a, q)\}$  are known to be  $q$ -holonomic [GS06, MM21], and the corresponding  $q$ -difference equation is known as the *quantum  $C$ -polynomial*.

The idea of inverted Habiro series [Par21] is to extend the sequence  $\{a_m(K; a, q)\}_{m \geq 0}$  to negative values of  $m$ . This can be often done by solving the quantum  $C$ -polynomial recursion in the negative direction. In particular, in [Par21] it was illustrated that  $a_m(\mathfrak{5}_2; q)$  for  $m < 0$  can be computed by recursively solving the  $q$ -difference equations and that

$$F_{\mathfrak{5}_2}(x, q) = F_{\mathfrak{5}_2}^{\mathfrak{sl}_2, (\text{ab})}(x, q) = - \sum_{m < 0} \frac{a_m(\mathfrak{5}_2; q)}{\prod_{0 \leq j \leq -m-1} (x + x^{-1} - q^j - q^{-j})}.$$

Using the  $a$ -deformed quantum  $C$ -polynomial for  $\mathfrak{5}_2$  knot given in [MM21], we can do the same for  $\mathfrak{sl}_N$ . For instance, the first few coefficients in the case of  $\mathfrak{sl}_3$  are given by

$$\begin{aligned} a_{-1}^{\mathfrak{sl}_3, (\text{ab})}(\mathfrak{5}_2; q) &= -q^{-2} + 1 - q^3 + q^7 - q^{12} + q^{18} - \dots, \\ a_{-2}^{\mathfrak{sl}_3, (\text{ab})}(\mathfrak{5}_2; q) &= q^{-3} - q^{-2} - q^{-1} + 1 + q + q^2 - q^3 - q^4 - q^5 - q^6 + q^7 + \dots, \\ a_{-3}^{\mathfrak{sl}_3, (\text{ab})}(\mathfrak{5}_2; q) &= -1 + q^2 + q^3 + q^4 - q^5 - q^6 - 2q^7 - q^8 + \dots, \end{aligned}$$

and so on. Combined with the following formula

$$F_K^{\mathfrak{sl}_N, (\text{ab})}(x, q) = - \sum_{n \geq 0} a_{-n-1}^{\mathfrak{sl}_N, (\text{ab})}(K; q) \frac{[-n]!}{\prod_{0 \leq j \leq n} (x^{\frac{1}{2}} q^{\frac{j}{2}} - x^{-\frac{1}{2}} q^{-\frac{j}{2}})(x^{\frac{1}{2}} q^{\frac{(N-2)-j}{2}} - x^{-\frac{1}{2}} q^{\frac{j-(N-2)}{2}})},$$

this allows us to compute  $F_{\mathfrak{5}_2}^{\text{sl}_N,(\text{ab})}(x, q)$  for any  $N$ .

Computation of  $F_{\mathfrak{5}_2}^{\text{sl}_N,(\infty)}(x, q)$  is similar. For this, we need to compute the inverted Habiro coefficients for the non-abelian branch  $(\infty)$ . This can be done by setting the initial condition to be

$$a_m^{\text{sl}_N,(\infty)}(\mathfrak{5}_2; q) = 0 \text{ for all } m \geq 0.$$

We find that, up to an overall factor,

$$\sum_{n \geq 0} a_{-n-1}^{\text{sl}_N,(\infty)}(\mathfrak{5}_2; q) E^{n+1} = E + \frac{1 + q^3 a^{-1}}{1 - q} E^2 + \frac{1 + q^4(1 + q)a^{-1} + q^8 a^{-2}}{(1 - q)(1 - q^2)} E^3 + \dots \Big|_{a=q^N},$$

where  $E$  is a formal variable. This implies, for instance, that up to an overall factor,

$$\begin{aligned} F_{\mathfrak{5}_2}^{\text{sl}_2,(\infty)}(x, q) &= x + \frac{3 - q}{1 - q} x^2 + \frac{q^{-1} + 7 - q - 4q^2 + q^3}{(1 - q)(1 - q^2)} x^3 + \dots, \\ F_{\mathfrak{5}_2}^{\text{sl}_3,(\infty)}(x, q) &= \frac{2}{1 - q} x^2 + \frac{5 + q - 2q^2}{(1 - q)^2} x^3 + \frac{q^{-1} + 10 + 7q - 3q^2 - 8q^3 - q^4 + 2q^5}{(1 - q)^2(1 - q^2)} x^4 + \dots \\ F_{\mathfrak{5}_2}^{\text{sl}_4,(\infty)}(x, q) &= \frac{3 + q}{(1 - q)(1 - q^2)} x^3 + \frac{7 + 3q + 2q^2 - 3q^3 - q^4}{(1 - q)^2(1 - q^2)} x^4 + \dots \end{aligned}$$

By computing the expectation value of the  $\hat{y}$ -operator (i.e.  $\lim_{q \rightarrow 1} \frac{F_K(qx, q)}{F_K(x, q)}$ ), it is easy to verify that these solutions are indeed associated to the non-abelian branch  $(\infty)$ .

**Remark 9.** If we set

$$\begin{aligned} f_0(b, q) &= \sum_{N \geq 2} f_0^{\text{sl}_N}(q) (q)_{N-2} b^{-N}, \\ f_1(b, q) &= \sum_{N \geq 2} f_1^{\text{sl}_N}(q) (1 - q) (q)_{N-2} b^{-N}, \end{aligned}$$

where  $b$  is a formal variable, and  $f_0^{\text{sl}_N}$  and  $f_1^{\text{sl}_N}$  denote the first and second coefficients of  $F_{\mathfrak{5}_2}^{\text{sl}_N,(\infty)}$  so that  $f_0^{\text{sl}_2}(q) = 1$ ,  $f_0^{\text{sl}_3}(q) = \frac{2}{1 - q}$ , and so on, then we find experimentally that they satisfy the following recurrence relations:

$$\begin{aligned} (\hat{b}^2 - 2\hat{b} + 1 - q^{-1}\hat{a})f_0(b, q) &= 1, \\ (1 - q)(\hat{b} - 1)f_1(b, q) + (2 - 2\hat{b} + q^{-1}(\hat{b} - 2)\hat{a})f_0(b, q) &= b^{-1}. \end{aligned}$$

Here,  $\hat{b}$  and  $\hat{a}$  are linear operators characterized by

$$\hat{b}b^{-N} = b^{-N+1}, \quad \hat{a}b^{-N} = q^N b^{-N}, \quad \hat{a}\hat{b} = q^{-1}\hat{b}\hat{a}.$$

Since the first two coefficients of  $F_{\mathfrak{5}_2}^{\text{sl}_N}$  completely determine the whole series via recursion, the above recurrence relations allow one to compute  $F_{\mathfrak{5}_2}^{\text{sl}_N,(\infty)}$  up to any desired order.

#### 4. $F_K$ INVARIANTS AND QUIVERS

In the previous section we have constructed  $F_K$  invariants for various branches using the quantum  $A$ -polynomials. In this section we will focus on the relation between these newly constructed  $F_K$  invariants and quivers.

**4.1. From knot conormal quivers to knot complement quivers.** We start by studying how we can obtain  $F_K$  invariants for some branches from the original quivers of [KRSS17, KRSS19] corresponding to knot conormals. Then we use it to construct quivers corresponding to some knot complements, generalizing [Kuc20]. Finally, we show that a slight modification of this construction leads to simpler quivers corresponding to the same  $F_K$  invariant.

The computation of  $F_K$  for abelian branches of left-handed  $(2, 2p + 1)$  torus knots in [EGG<sup>+</sup>20] relies on the fact that there exists a simple Fourier transform between coloured HOMFLY-PT polynomials  $P_r(a, q)$  and  $F_K(x, a, q)$ , which is essentially a substitution  $x = q^r$ . However, this does not work in general and  $F_K$  cannot be obtained directly from the knowledge of  $P_r$ .

One way to deal with this problem is to consider a knot  $K$  with a suitable framing  $f$ , so that after replacing  $q^r = x$ , the corresponding coloured HOMFLY-PT polynomials  $P_r(a, q)$  become a power series in  $x$ . In order to find the correct framing and to compute the corresponding power series, the conormal quivers become crucially important. First of all, the framing will be the absolute value of the minimal entry of the conormal quiver matrix. Moreover, the power series in  $x$  can be quickly determined and will be given in the quiver form. Finally, the obtained power series will be equal to  $F_K$  invariant for the branch corresponding to the smallest slope of the knot  $K$ . Therefore, for each knot,  $F_K$  for one branch can be obtained by this procedure. In particular, this branch will be abelian only when the corresponding framing, i.e. the smallest entry of the conormal quiver matrix, is equal to zero. In all other cases (like figure-eight knot, right-handed trefoil, etc.), the procedure that we outline below will produce  $F_K$  corresponding to a certain non-abelian branch.

Now let us pass to details of the connection between conormal quivers,  $F_K$  invariants and their quiver forms. Let  $K$  be a knot with a corresponding quiver  $Q$  with  $m$  vertices and the adjacency matrix  $C_{ij}$ . Also let  $\mathbf{a}$  and  $\mathbf{l}$  be the vectors corresponding to the linear terms and essentially to uncoloured polynomial. Then

$$P_r(K; a, q) = \sum_{d_1 + \dots + d_k = r} (-1)^{\sum C_{ii}d_i} a^{\sum a_i d_i} q^{\sum l_i d_i} q^{\frac{1}{2} \sum_{i,j=1}^m C_{ij}d_i d_j} \frac{(q)_r}{\prod_{i=1}^k (q)_{d_i}}.$$

Now suppose that

$$-C_{\min} \leq C_{ij} \leq C_{\max}, \quad i, j = 1, \dots, m,$$

where  $C_{\min}, C_{\max} \geq 0$  (must be since  $C_{kk} = 0$ , for some  $k$ ), and permute rows and columns of  $C$  such that  $C_{11} = C_{\min}$  and  $C_{mm} = C_{\max}$ . Note that in the cases where conormal quivers have been computed, the largest and smallest entries are on the diagonal.

Then for knots which allow for a simple Fourier transform between  $P_r$  and  $F_K$  (that boils down to substitution  $x = q^r$ ), we can obtain  $F_K$  invariant for  $C_{\min}$ -framed knot  $K$  by multiplying each  $P_r$  by  $q^{\frac{1}{2}C_{\min}(r^2-r)}$ :

$$\begin{aligned} F_{K^{f=C_{\min}}}(x, a, q) &= (-1)^{rC_{\min}} a^{ra_1} q^{rq_1} \sum_{d_2, \dots, d_m} (-1)^{\sum_{i \geq 2} (C_{ii} + C_{\min})d_i} a^{\sum_{i \geq 2} (a_i - a_1)d_i} \\ (30) \quad &\times q^{\sum_{i \geq 2} (l_i - l_1)d_i} x^{\sum_{i \geq 2} (C_{1i} + C_{\min})d_i} q^{\frac{1}{2} \sum_{i,j \geq 2} (C_{ij} - C_{i1} - C_{1j} + C_{11})d_i d_j} \\ &\times \frac{(x; q^{-1})_{d_2 + \dots + d_k}}{\prod_{i=2}^k (q)_{d_i}}. \end{aligned}$$

For the mirror image of  $K$ , the quiver and the change of variables are given by

$$C_{m(K)} = -C_K + I_{m \times m} - J_{m \times m}, \quad \mathbf{a}_{m(K)} = -\mathbf{a}_K, \quad \mathbf{l}_{m(K)} = -\mathbf{l}_K,$$

where  $J_{m \times m}$  is the  $m \times m$  matrix with all entries equal to 1. Then the diagonal entries of  $C_{m(K)}$  are bigger than  $-C_{\max}$  and smaller than  $C_{\min}$ , with  $(C_{m(K)})_{11} = C_{\min}$  and  $(C_{m(K)})_{mm} = -C_{\max}$ . In consequence, we can apply the above procedure for the  $f = C_{\max}$  framing of knot  $m(K)$ :

$$\begin{aligned}
(31) \quad F_{(m(K))f=C_{\max}}(x, a, q) &= \sum_{d_1, \dots, d_{m-1}} (-1)^{\sum_i (C_{ii} + C_{\max})d_i} a^{\sum_i (a_m - a_i)d_i} \\
&\times q^{\sum_i (l_m - l_i)d_i} x^{\sum_i (C_{\max} - C_{im} - 1)d_i} q^{-\frac{1}{2} \sum_{i,j} C_{ij}d_i d_j} \\
&\times q^{\sum_i C_{im}d_i \sum_i d_i} q^{-\frac{1}{2} C_{\max} (\sum_i d_i)^2} q^{-\sum_{i < j} d_i d_j} \frac{(x; q^{-1})_{d_1 + \dots + d_{m-1}}}{\prod_{i=1}^{m-1} (q)_{d_i}}.
\end{aligned}$$

Equations (30) and (31) are very close to the quiver form. The simplest way of reaching it is an application of Lemma 4.5 from [KRSS19]:

$$\begin{aligned}
(32) \quad \frac{(x; q^{-1})_{d_1 + \dots + d_n}}{(q)_{d_1} \cdots (q)_{d_n}} &= \frac{(x q^{1 - \sum_i d_i}; q)_{d_1 + \dots + d_n}}{(q)_{d_1} \cdots (q)_{d_n}} \\
&= \sum_{\alpha_1 + \beta_1 = d_1} \cdots \sum_{\alpha_n + \beta_n = d_n} (-q^{1/2})^{\beta_1^2 + \dots + \beta_n^2 + 2 \sum_{i=1}^{n-1} \beta_{i+1} (d_1 + \dots + d_i)} \\
&\quad \times \frac{(x q^{1/2 - \sum_i \alpha_i - \sum_i \beta_i})^{\beta_1 + \dots + \beta_n}}{(q)_{\alpha_1} (q)_{\beta_1} \cdots (q)_{\alpha_n} (q)_{\beta_n}}.
\end{aligned}$$

In the next section we will show in examples that this expansion leads to expressions for knot complement quivers found in [Kuc20]. On the other hand, we can use the following formula:

$$(33) \quad (x; q^{-1})_d = (x q^{1-d}; q)_d = \frac{(x q^{1-d}; q)_{\infty}}{(x q; q)_{\infty}} = \sum_{i,j} (-1)^i x^{i+j} q^{i+j} q^{-di} q^{\frac{1}{2}(i^2-i)} \frac{1}{(q)_i (q)_j}.$$

Since it effectively adds two nodes instead of doubling them, resulting quivers are expected to be simpler. In the next section we will see in examples that it is indeed the case. We will also see that the transition between two kinds of quivers can be interpreted in terms of linking and removing a redundant pair of nodes, defined in Section 2.4 (for details see [EKL20a]).

In addition to the reasoning presented above, quivers corresponding to knot complements can be often obtained directly in the simpler form by matching quiver adjacency matrix and the change of variables against order by order expansion of  $F_K$  invariants for various branches (which can be obtained from  $A$ -polynomials, as we saw in Section 3). Many results presented in the next section were derived in this way.

**4.2. Examples.** In this section we illustrate the considerations presented above on the example of the figure-eight and trefoil knots, taking into account  $F_K$  and  $F_{m(K)}$  for various branches. Moreover, the application of methods from Section 4.1 to the results of [Kuc20] enables us to conjecture the simple quiver form for general  $(2, 2p+1)$  torus. Analogous results for all knots with 5 or 6 crossings, as well as  $(3,4)$  torus knot, are presented in Appendix A.

**4.2.1. Figure-eight.** First we shall obtain the quiver for the slope  $-\frac{1}{2}$  non-abelian branch of the figure-eight knot, following the discussion from Section 4.1. We start from the expression for the coloured HOMFLY-PT polynomial for  $\mathbf{4}_1$  obtained in [KRSS19]:

$$(34) \quad P_r(\mathbf{4}_1; a, q) = \sum_{\tilde{d}_1 + \dots + \tilde{d}_5 = r} (-1)^{\tilde{d}_3 + \tilde{d}_4} a^{\tilde{d}_2 - \tilde{d}_5} q^{-\tilde{d}_2 - \frac{1}{2}\tilde{d}_3 + \frac{1}{2}\tilde{d}_4 + \tilde{d}_5} q^{\frac{1}{2} \sum_{i,j=1}^5 C_{ij} \tilde{d}_i \tilde{d}_j} \frac{(q; q)_r}{\prod_{i=1}^5 (q; q)_{\tilde{d}_i}},$$

where

$$C = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 & -1 \\ -1 & -1 & -2 & -1 & -2 \end{pmatrix}.$$

We need to add framing  $f = 2$ , i.e. to multiply by  $q^{r(r-1)}$ . Since

$$C + 2 \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

has 0 at the bottom right corner and all entries in the last row are non-negative, we shall replace  $\tilde{d}_5 = r - \tilde{d}_1 - \tilde{d}_2 - \tilde{d}_3 - \tilde{d}_4$  in (34), which leads to

$$\begin{aligned} q^{r(r-1)} P_r(\mathbf{4}_1; a, q) &= \sum_{\tilde{d}_1 + \cdots + \tilde{d}_4 \leq r} (-1)^{\tilde{d}_3 + \tilde{d}_4} a^{-r} a^{\tilde{d}_1 + 2\tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4} q^{-\tilde{d}_1 - 2\tilde{d}_2 - \frac{3}{2}\tilde{d}_3 - \frac{1}{2}\tilde{d}_4} \frac{(q^r; q^{-1})_{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4}}{\prod_{i=1}^4 (q; q)_{\tilde{d}_i}} \\ &\quad \times q^{\frac{1}{2}\sum_{i,j=1}^4 C_{ij}\tilde{d}_i\tilde{d}_j - (\tilde{d}_1 + \tilde{d}_2 + 2\tilde{d}_3 + \tilde{d}_4)(r - \tilde{d}_1 - \tilde{d}_2 - \tilde{d}_3 - \tilde{d}_4) - (r - \tilde{d}_1 - \tilde{d}_2 - \tilde{d}_3 - \tilde{d}_4)^2 + r^2} \\ &= a^{-r} \sum_{\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4 \leq r} (-1)^{\tilde{d}_3 + \tilde{d}_4} a^{\tilde{d}_1 + 2\tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4} q^{-\tilde{d}_1 - 2\tilde{d}_2 - \frac{3}{2}\tilde{d}_3 - \frac{1}{2}\tilde{d}_4} q^{r(\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_4)} \\ &\quad \times q^{\frac{1}{2}\sum_{i,j=1}^4 \tilde{C}_{ij}\tilde{d}_i\tilde{d}_j} \frac{(q^r; q^{-1})_{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4}}{\prod_{i=1}^4 (q; q)_{\tilde{d}_i}} \end{aligned}$$

Performing the substitution  $q^r \rightarrow x$ , we obtain – up to an overall prefactor – the following expression:

$$\begin{aligned} F_{4_1}^{(-\frac{1}{2})}(x, a, q) &= \sum_{\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4 \geq 0} (-1)^{\tilde{d}_3 + \tilde{d}_4} a^{\tilde{d}_1 + 2\tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4} q^{-\tilde{d}_1 - 2\tilde{d}_2 - \frac{3\tilde{d}_3 + \tilde{d}_4}{2}} q^{\frac{1}{2}\sum_{i,j=1}^4 \tilde{C}_{ij}\tilde{d}_i\tilde{d}_j} \\ &\quad \times x^{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_4} \frac{(x; q^{-1})_{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4}}{\prod_{i=1}^4 (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

In order to obtain  $F_K$  in a quiver form, we need to expand the  $q$ -Pochhammer  $(x; q^{-1})_{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4}$ . We can do that in either of two ways described in Section 4.1.

If we use (32), we get

$$\begin{aligned} \frac{(x; q^{-1})_{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4}}{\prod_{i=1}^4 (q)_{\tilde{d}_i}} &= \sum_{\alpha_1 + \beta_1 = \tilde{d}_1} \cdots \sum_{\alpha_4 + \beta_4 = \tilde{d}_4} (-q^{1/2})^{\beta_1^2 + \cdots + \beta_4^2 + 2\sum_{i=1}^3 \beta_{i+1}(\tilde{d}_1 + \cdots + \tilde{d}_i)} \\ &\quad \times \frac{(x q^{1/2 - \sum_i \alpha_i - \sum_i \beta_i})^{\beta_1 + \cdots + \beta_4}}{(q)_{\alpha_1} (q)_{\beta_1} \cdots (q)_{\alpha_4} (q)_{\beta_4}}. \end{aligned}$$

In such a way we have eight summation variables  $\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4$ , satisfying  $\tilde{d}_i = \alpha_i + \beta_i$ ,  $i = 1, \dots, 4$ . Focusing on the powers of  $q$  that are quadratic in these summation variables, it can be seen from the formula above that the corresponding matrix  $C$  is of the form

$$C = \left( \begin{array}{c|c} \tilde{C} & \tilde{C} - L \\ \hline \tilde{C} - U & \tilde{C} - J \end{array} \right),$$

where  $J$  is matrix of all ones, and  $L$  and  $U$  are lower and upper triangular matrix, respectively, consisting only of ones:

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Therefore, with summation variables  $(d_1, \dots, d_8) = (\alpha_1, \dots, \alpha_4, \beta_1, \dots, \beta_4)$ , we get

$$(35) \quad F_{4_1}^{(-\frac{1}{2})}(x, a, q) = \sum_{d_1, \dots, d_8 \geq 0} (-q^{\frac{1}{2}})^{\sum_{i,j=1}^8 C_{ij} d_i d_j} \prod_{i=1}^8 \frac{x_i^{d_i}}{(q)_{d_i}},$$

with

$$(36) \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} xaq^{-1} \\ xa^2q^{-2} \\ aq^{-\frac{3}{2}} \\ xaq^{-\frac{1}{2}} \\ x^2aq^{-\frac{1}{2}} \\ x^2a^2q^{-\frac{3}{2}} \\ xaq^{-1} \\ x^2a \end{pmatrix}.$$

Alternatively, if we use the expansion (33), we get

$$(x; q^{-1})_{d_1+\dots+d_4} = \sum_{i,j \geq 0} (-1)^i x^{i+j} q^{\frac{1}{2}i+j} q^{-i(d_1+d_2+d_3+d_4)} q^{\frac{1}{2}i^2} \frac{1}{(q)_i (q)_j}.$$

Hence, in this case we get an analogue of (35) with six summation variables  $(d_1, \dots, d_6) = (\tilde{d}_1, \dots, \tilde{d}_4, i, j)$  and with the quiver matrix  $C$  and vector  $\mathbf{x}$  given by

$$(37) \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} xaq^{-1} \\ xa^2q^{-2} \\ aq^{-\frac{3}{2}} \\ xaq^{-\frac{1}{2}} \\ xq^{\frac{1}{2}} \\ xq \end{pmatrix}.$$

Let us move to the abelian branch. In that case we cannot apply the reasoning from Section 4.1 since some entries of conormal quiver are negative. In particular, we have  $C_{\min} = -2$ , so framing 2 would be needed, as we saw in the paragraph above. However, for

the abelian branch we can use a direct approach, matching quiver adjacency matrix and the change of variables against order by order expansion of  $F_{4_1}$ . This leads to

$$(38) \quad F_{4_1}(x, a, q) = \sum_{d_1, \dots, d_6 \geq 0} (-q^{\frac{1}{2}})^{\sum_{i,j=1}^6 C_{ij} d_i d_j} \prod_{i=1}^6 \frac{x_i^{d_i}}{(q)_{d_i}}$$

with

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} qx \\ qx \\ qx \\ q^{-\frac{1}{2}}ax \\ q^{-\frac{1}{2}}ax \\ q^{-\frac{1}{2}}ax \end{pmatrix},$$

and  $C$  given by any of the following matrices:

$$(39) \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Note that all these matrices are equivalent in the sense of [JKL+21].

Analogous approach applied to the non-abelian branch with slope  $-\frac{1}{2}$  leads to

$$(40) \quad F_{4_1}^{(-\frac{1}{2})}(x, a, q) = \sum_{d_1, \dots, d_5 \geq 0} (-q^{\frac{1}{2}})^{\sum_{1 \leq i, j \leq 5} C_{ij} d_i d_j} \prod_{i=1}^5 \frac{x_i^{d_i}}{(q)_{d_i}}$$

with

$$(41) \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} qx \\ ax \\ q^{\frac{3}{2}}x \\ q^{\frac{3}{2}}x \\ q^{-\frac{1}{2}}ax \end{pmatrix}.$$

We expect this quiver along with the quivers (36) and (37) are all equivalent (up to a factor independent of  $x$ ).

4.2.2. *Trefoil*. In the case of the abelian branch of the left-handed trefoil, we can use the formula given in [EGG+20]:

$$(42) \quad F_{3_1}(x, a, q) = \sum_{k \geq 0} x^k q^k \frac{(x; q^{-1})_k (aq^{-1}; q)_k}{(q)_k}.$$

Expanding  $(aq^{-1}; q)_k$  using Lemma 4.5 from [KRSS19], we get an expression analogous to Equation (30):

$$F_{3_1}(x, a, q) = \sum_{\tilde{d}_1, \tilde{d}_2 \geq 0} (-1)^{\tilde{d}_1} q^{\tilde{d}_2} x^{\tilde{d}_1 + \tilde{d}_2} a^{\tilde{d}_1} q^{\frac{1}{2}(\tilde{d}_1^2 - \tilde{d}_1)} \frac{(x; q^{-1})_{\tilde{d}_1 + \tilde{d}_2}}{(q)_{\tilde{d}_1} (q)_{\tilde{d}_2}}.$$

As explained in Section 4.1, we can expand the last fraction in two ways. Simpler expansion (33) leads to

$$(43) \quad F_{\mathbf{3}_1}(x, a, q) = \sum_{d_1, \dots, d_4 \geq 0} (-q^{\frac{1}{2}})^{\sum_{i,j=1}^4 C_{ij} d_i d_j} \prod_{i=1}^4 \frac{x_i^{d_i}}{(q)_{d_i}},$$

where

$$(44) \quad C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} xq \\ xaq^{-1/2} \\ xq \\ xq^{1/2} \end{pmatrix}.$$

On the other hand, the expansion (32) gives (43) with the quiver

$$(45) \quad C = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} xq \\ xaq^{-1/2} \\ x^2q^{3/2} \\ x^2a \end{pmatrix},$$

which reproduces the result of [Kuc20].

We can connect (44) with (45) directly, using the procedures of linking and removing a redundant pair of nodes, defined in Section 2.4 (for details see [EKL20a]). We start from (44) and link nodes number 2 and 4, which leads to the quiver

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} xq \\ xaq^{-1/2} \\ xq \\ xq^{1/2} \\ x^2a \end{pmatrix}.$$

Then linking nodes number 1 and 4 gives

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} xq \\ xaq^{-1/2} \\ xq \\ xq^{1/2} \\ x^2a \\ x^2q^{3/2} \end{pmatrix}.$$

Now we can notice that nodes number 3 and 4 form a redundant pair. Removing it leads to

$$C = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} xq \\ xaq^{-1/2} \\ x^2a \\ x^2q^{3/2} \end{pmatrix},$$

which – after exchanging the last two nodes – is equal to quiver (45).

For the mirror image of the left-handed trefoil (i.e. the right-handed trefoil) in framing  $f = 3$ , the prescription from Section 4.1 leads to the following formula for the non-abelian

branch with slope  $\frac{1}{3}$ :

$$(46) \quad F_{\mathbf{3}_1^r}^{(\frac{1}{3})}(x, a, q) = \sum_{\tilde{d}_1, \tilde{d}_2 \geq 0} (-1)^{\tilde{d}_1 + \tilde{d}_2} a^{\tilde{d}_1 + \tilde{d}_2} q^{\frac{1}{2}\tilde{d}_1^2 - \frac{1}{2}\tilde{d}_1 + \frac{1}{2}\tilde{d}_2^2 - \frac{3}{2}\tilde{d}_2} x^{\tilde{d}_1} \frac{(x; q^{-1})_{\tilde{d}_1 + \tilde{d}_2}}{(q)_{\tilde{d}_1} (q)_{\tilde{d}_2}}.$$

Then we rewrite the  $q$ -Pochhammer  $(x; q^{-1})_{\tilde{d}_1 + \tilde{d}_2}$  using expansion (33):

$$(47) \quad F_{\mathbf{3}_1^r}^{(\frac{1}{3})}(x, a, q) = \sum_{\tilde{d}_1, \tilde{d}_2, i, j} (-1)^{\tilde{d}_1 + \tilde{d}_2 + i} a^{\tilde{d}_1 + \tilde{d}_2} q^{\frac{1}{2}(\tilde{d}_1^2 - \tilde{d}_1) + \frac{1}{2}(\tilde{d}_2^2 - \tilde{d}_2)} \frac{x^{\tilde{d}_1 + i + j} q^{i + j - \tilde{d}_2} q^{-i\tilde{d}_1 - i\tilde{d}_2} q^{\frac{1}{2}(i^2 - i)}}{(q)_{\tilde{d}_1} (q)_{\tilde{d}_2} (q)_i (q)_j}.$$

We can sum over index  $\tilde{d}_2$  that does not appear in the power of  $x$ :

$$(48) \quad \sum_{\tilde{d}_2} (-1)^{\tilde{d}_2} a^{\tilde{d}_2} q^{-\tilde{d}_2} q^{-i\tilde{d}_2} q^{\frac{1}{2}(\tilde{d}_2^2 - \tilde{d}_2)} \frac{1}{(q)_{\tilde{d}_2}} = (aq^{-1-i}; q)_\infty = (aq^{-1}; q)_\infty (aq^{-1-i}; q)_i$$

and use the formula

$$(49) \quad \frac{(aq^{-1-i}; q)_i}{(q)_i} = \sum_{\alpha + \beta = i} (-1)^\alpha a^\alpha q^{-\alpha - i\alpha} q^{\frac{1}{2}(\alpha^2 - \alpha)} \frac{1}{(q)_\alpha (q)_\beta}.$$

Substituting (48-49) back in (47), we obtain

$$F_{\mathbf{3}_1^r}^{(\frac{1}{3})}(x, a, q) = (aq^{-1}; q)_\infty \sum_{\tilde{d}_1, \alpha, \beta, j} (-1)^{\tilde{d}_1 + \beta} a^{\tilde{d}_1 + \alpha} q^{j + \beta - \alpha} \frac{q^{\frac{1}{2}(\tilde{d}_1^2 - \tilde{d}_1) + \frac{1}{2}(\beta^2 - \beta) - \tilde{d}_1\alpha - \tilde{d}_1\beta} x^{\tilde{d}_1 + \alpha + \beta + j}}{(q)_{\tilde{d}_1} (q)_\alpha (q)_\beta (q)_j}.$$

Finally, after dividing by the overall infinite  $q$ -Pochhammer independent of  $x$ , we get the quiver form

$$(50) \quad F_{\mathbf{3}_1^r}^{(\frac{1}{3})}(x, a, q) = \sum_{d_1, d_2, d_3, d_4 \geq 0} (-q^{\frac{1}{2}})^{\sum_{1 \leq i, j \leq 4} C_{ij} d_i d_j} \prod_{i=1}^4 \frac{x_i^{d_i}}{(q)_{d_i}}$$

with

$$(51) \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} xq \\ xaq^{-1} \\ xaq^{-\frac{1}{2}} \\ xq^{\frac{1}{2}} \end{pmatrix}.$$

In the case of the right-handed trefoil and abelian branch, we cannot apply the reasoning from Section 4.1 because it does not correspond to framing  $f = C_{\min}$  nor  $f = C_{\max}$ . However, similarly to the abelian branch of figure-eight knot, we can match quiver adjacency matrix and the change of variables against order by order expansion of  $F_{\mathbf{3}_1^r}$ , which leads to

$$(52) \quad F_{\mathbf{3}_1^r}(x, a, q) = \sum_{d_1, d_2, d_3, d_4 \geq 0} (-q^{\frac{1}{2}})^{\sum_{1 \leq i, j \leq 4} C_{ij} d_i d_j} \prod_{i=1}^4 \frac{x_i^{d_i}}{(q)_{d_i}}$$

with

$$(53) \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} xq \\ xa \\ xaq^{-\frac{1}{2}} \\ xaq^{-\frac{1}{2}} \end{pmatrix}.$$

4.2.3.  $(2, 2p + 1)$  torus knots. In [Kuc20] it was shown that there is a recursion relating quivers corresponding to  $(2, 2p + 1)$  torus knot complements. Those quivers are obtained by expanding  $q$ -Pochhammer  $(x; q^{-1})_d$  in the general formula from [EGG<sup>+</sup>20] via (32). Using expansion (33), we can obtain the corresponding quivers in an even simpler form.

Let us start our analysis from a slightly rearranged form of (44):

$$F_{\mathfrak{3}_1}(x, a, q) = \sum_{d_1, \dots, d_4 \geq 0} (-q^{1/2})^{d \cdot C \cdot d} \frac{x^{\mathbf{n} \cdot d} a^{\mathbf{a} \cdot d} q^{d \cdot \mathbf{q} \cdot d - \frac{1}{2} \sum_i C_{ii} d_i}}{(q)_d},$$

where  $q_i - \frac{1}{2} C_{ii} = l_i$  and the vectors  $\mathbf{n}, \mathbf{a}, \mathbf{q}$  and the matrix  $C$  are given by

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{aligned} \mathbf{n} &= (1, 1, 1, 1), \\ \mathbf{a} &= (1, 0, 0, 0), \\ \mathbf{q} &= (0, 1, 1, 1) = 1 - \mathbf{a}. \end{aligned}$$

The key idea is to keep in mind that the upper-left  $2 \times 2$  matrix comes from  $q^{\frac{1}{2}d_1^2}$  in (42), whereas the last two rows and columns originate from the expansion of the  $q$ -Pochhammer  $(x, q^{-1})_{d_1+d_2}$ . For more complicated torus knots, the upper-left part will grow following the quadratic  $q$  powers in the analogue of (42), whereas the last two rows and columns remain unchanged. Let us see it in the example of  $\mathfrak{5}_1$  and  $\mathfrak{7}_1$  knots.

For the  $\mathfrak{5}_1$  knot we can derive

$$(54) \quad F_{\mathfrak{5}_1}(x, a, q) = \sum_{d_1, \dots, d_6 \geq 0} (-q^{1/2})^{d \cdot C \cdot d} \frac{x^{\mathbf{n} \cdot d} a^{\mathbf{a} \cdot d} q^{d \cdot \mathbf{q} \cdot d - \frac{1}{2} \sum_i C_{ii} d_i}}{(q)_d},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -2 & -1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{aligned} \mathbf{n} &= (1, 1, 3, 3, 1, 1), \\ \mathbf{a} &= (1, 0, 1, 0, 0, 0), \\ \mathbf{q} &= (0, 1, 0, 1, 1, 1) = 1 - \mathbf{a}, \end{aligned}$$

whereas for  $\mathfrak{7}_1$  we have

$$F_{\mathfrak{7}_1}(x, a, q) = \sum_{d_1, \dots, d_8 \geq 0} (-q^{1/2})^{d \cdot C \cdot d} \frac{x^{\mathbf{n} \cdot d} a^{\mathbf{a} \cdot d} q^{d \cdot \mathbf{q} \cdot d - \frac{1}{2} \sum_i C_{ii} d_i}}{(q)_d},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -2 & -2 & -2 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 & -3 & -1 & 0 \\ 0 & -1 & -2 & -3 & -3 & -4 & -1 & 0 \\ 0 & -1 & -2 & -3 & -4 & -4 & -1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{aligned} \mathbf{n} &= (1, 1, 3, 3, 5, 5, 1, 1), \\ \mathbf{a} &= (1, 0, 1, 0, 1, 0, 0, 0), \\ \mathbf{q} &= (0, 1, 0, 1, 0, 1, 1, 1) \\ &= 1 - \mathbf{a}. \end{aligned}$$

Using the general formula from [EGG<sup>+</sup>20], we can show that the pattern continues and

$$F_{T_{2,2p+1}}(x, a, q) = \sum_{d_1, \dots, d_{2p+2} \geq 0} (-q^{1/2})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} a^{\mathbf{a} \cdot \mathbf{d}} q^{\mathbf{q} \cdot \mathbf{d} - \frac{1}{2} \sum_i C_{ii} d_i}}{(q)_{\mathbf{d}}}$$

$$C = \begin{pmatrix} \mathbf{I}_{2p} - \mathbf{D} & -\mathbf{1} & \mathbf{0} \\ -\mathbf{1} & 1 & 0 \\ \mathbf{0} & 0 & 0 \end{pmatrix}, \quad \begin{aligned} \mathbf{n} &= (1, 1, 3, 3, \dots, 2p-1, 2p-1, 1, 1), \\ \mathbf{a} &= (1, 0, \dots, 1, 0, 0, 0), \\ \mathbf{q} &= (0, 1, \dots, 0, 1, 1, 1) = \mathbf{1} - \mathbf{a}, \end{aligned}$$

where  $-\mathbf{1}, \mathbf{0}$  denote constant vectors of appropriate size,  $\mathbf{I}_{2p}$  is the identity matrix and  $\mathbf{D}$  is the matrix  $D_{ij} = \min(i, j) - 1$  with  $1 \leq i, j \leq 2p$ . Note that we always have a totally disconnected node which we can remove and replace with a  $q$ -Pochhammer prefactor.

**4.3.  $F_K$  invariants and quivers from classical  $A$ -polynomials and branches.** In the previous sections we have made two interesting discoveries. First, we associated  $F_K$  invariants to various branches of  $A$ -polynomials. Second, we found that these  $F_K$  invariants can be encoded in quiver generating series, analogously as in the knots-quivers correspondence [KRSS17, KRSS19]. Let us now stress that these two ideas are intimately related. On one hand, the branches  $y = y(x)$  of  $A$ -polynomials can be obviously determined from the quiver form (or any other form) of  $F_K$  invariants. On the other hand, more interestingly, it turns out that just the existence of the quiver form imposes strong constraints, which enable to determine an underlying quiver simply from the finite number of terms in the expansion  $y = y(x)$  for a given branch. In this section we illustrate these relations; note that similar reconstruction methods were discussed in [PSS18, BJS20]. It would also be desirable to understand better which property of  $A$ -polynomials asserts the existence of the quiver form of the corresponding  $F_K$  invariants; presumably, it might be related to K-theoretic conditions that  $A$ -polynomials must meet too [Guk05, GS12b].

To proceed, let us again write down the quiver generating function (18):

$$(55) \quad P_Q(x_1, x_2, \dots, x_m, q) = \sum_{d_1, \dots, d_m=0}^{\infty} (-q^{1/2})^{\sum_{i,j=1}^m C_{ij} d_i d_j} \frac{x_1^{d_1} \cdots x_m^{d_m}}{(q)_{d_1} \cdots (q)_{d_m}}.$$

To such a series we can associate a classical expansion

$$y(x_1, \dots, x_m) = \lim_{q \rightarrow 1} \frac{P_Q(qx_1, qx_2, \dots, qx_m, q)}{P_Q(x_1, x_2, \dots, x_m, q)} \equiv \sum_{k_1, \dots, k_m} b_{k_1, \dots, k_m} x_1^{k_1} \cdots x_m^{k_m}.$$

In [PSS18] general expressions for coefficients  $b_{k_1, \dots, k_m}$  (as well as for numerical Donaldson-Thomas invariants that arise from factorization of the above series) in terms of elements of the quiver matrix  $C$  have been found. Currently we are interested in quiver generating series  $F(x, a, q)$  and corresponding  $y(x) = \sum_i y_i x^i$ , which depend on a single generating parameter  $x$ , and arise from specialization  $x_i = (-1)^{\#} a^{\#} q^{\#} x^{\#}$  of the above expressions. In principle, coefficients  $y_i$  can be obtained from  $b_{k_1, \dots, k_m}$ , however such expressions are quite complicated. It is therefore more convenient to determine coefficients  $y_i$  directly, having specialized  $x_i$  first.

Therefore, for the purpose of our discussion and brevity, consider the specialization such that each  $x_i$  is proportional to a single power of  $x$  (and also fix  $t = -1$ ), so that  $x_i = a^{a_i} q^{l_i} x$ . In this case  $F(x, a, q) = P_Q(\{x_i = a^{a_i} q^{l_i} x\})$  and we are interested in the corresponding

classical series:

$$(56) \quad y(x) = \lim_{q \rightarrow 1} \frac{F(qx, a, q)}{F(x, a, q)} \equiv \sum_{i=0}^{\infty} y_i x^i.$$

Note that linear powers of  $q$  in  $F(x, a, q)$  (that arise from specialization of  $x_i$ ) do not play a role at the classical level, so the coefficients  $y_i$  depend only on entries of  $C$  and the vector  $\mathbf{a}$ . Taking the above limit explicitly we find coefficients  $y_i$ :

$$(57) \quad \begin{aligned} y_1 &= - \sum_{i=1}^m (-1)^{C_{ii}} a^{a_i}, \\ y_2 &= \sum_{i=1}^m C_{ii} a^{2a_i} + \sum_{i < j} (-1)^{C_{ii} + C_{jj}} a^{a_i + a_j} (1 + 2C_{ij}), \\ y_3 &= -\frac{1}{2} \left( \sum_{i=1}^m (-1)^{C_{ii}} a^{3a_i} C_{ii} (3C_{ii} - 1) + \sum_{i \neq j} (-1)^{C_{jj}} a^{2a_i + a_j} (2C_{ii} + C_{ij}) (3C_{ij} + 1) \right. \\ &\quad \left. + \sum_{i < j < k} (-1)^{C_{ii} + C_{jj} + C_{kk}} a^{a_i + a_j + a_k} (2 + 4C_{jk} + C_{ik} (4 + 6C_{jk}) + C_{ij} (4 + 6C_{ik} + 6C_{jk})) \right). \end{aligned}$$

For a quiver of size  $m = 2$ , denoting

$$C_{1,1} = \alpha, \quad C_{1,2} = C_{2,1} = \beta, \quad C_{2,2} = \gamma,$$

and keeping parameters  $a_1, a_2$  and  $q_1, q_2$ , let us also write down one more coefficient explicitly:

$$(58) \quad \begin{aligned} y_1^{m=2} &= -(-1)^\alpha a^{a_1} - (-1)^\gamma a^{a_2}, \\ y_2^{m=2} &= \alpha a^{2a_1} + \gamma a^{2a_2} + (-1)^{\alpha+\gamma} a^{a_1+a_2} (1 + 2\beta), \\ y_3^{m=2} &= -\frac{1}{2} \left( (-1)^\alpha a^{3a_1} \alpha (3\alpha - 1) + (-1)^\gamma a^{3a_2} \gamma (3\gamma - 1) \right. \\ &\quad \left. + (-1)^\gamma a^{2a_1+a_2} (2\alpha + \beta) (3\beta + 1) + (-1)^\alpha a^{2a_2+a_1} (2\gamma + \beta) (3\beta + 1) \right), \\ y_4^{m=2} &= -\frac{1}{6} \left( 2a^{4a_1} \alpha (2\alpha - 1) (4\alpha - 1) + 2a^{4a_2} \gamma (2\gamma - 1) (4\gamma - 1) \right. \\ &\quad + (-1)^{\alpha+\gamma} a^{3a_1+a_2} (3\alpha + \beta) (3\alpha + \beta - 1) (4\beta + 1) \\ &\quad + (-1)^{\alpha+\gamma} a^{3a_2+a_1} (3\gamma + \beta) (3\gamma + \beta - 1) (4\beta + 1) \\ &\quad \left. + 6a^{2a_1+2a_2} (\alpha + \beta) (\beta + \gamma) (4\beta + 1) \right). \end{aligned}$$

Clearly, once we know the quiver matrix  $C$  and powers  $a_i$ , we can immediately determine coefficients  $y_i$  in the classical series  $y(x)$ . However, we can also consider the opposite perspective, and reconstruct the quiver matrix  $C$  and coefficients  $a_i$ , by comparing explicit form of (a finite number of) coefficients  $y_i$  with the above formulas. In particular, this is the situation we have to deal with when considering various branches of  $A$ -polynomial. Each branch is associated to one solution  $y = y(x)$  of the ( $a$ -deformed)  $A$ -polynomial equation  $A(x, y, a) = 0$ . If we assume that for each branch there exists a  $q$ -series  $F(x, a, q)$  that has a quiver representation, then (for each branch) we can determine the corresponding  $C$  and  $a_i$ . In addition, parameters  $q_i$  can be determined by comparison with a few first terms of the quantum series (55), if only we can determine them by other means.

4.3.1. *Examples: trefoil and figure-eight.* Let us illustrate the above relations on the example of trefoil and figure-eight knot (still, for simplicity, restricting to  $t = -1$  case). For the abelian branch of right-handed trefoil, the generating function (52) is determined by the quiver matrix  $C$  of size  $m = 4$ , given in (53), and  $\mathbf{a} = (a_1, a_2, a_3, a_4) = (0, 1, 1, 1)$ , so that (also taking appropriate  $l_i$  into account):

$$F(x, a, q) = 1 + x \frac{(a - q)}{q - 1} + x^2 \frac{(a^2(-q^2 + q + 1) + aq(q^2 - q - 2) + q^2)}{(q - 1)^2(q + 1)} + \dots$$

The classical  $y(x)$  can be determined either by solving the  $A$ -polynomial equation  $A(x, y) = 0$ , or by taking the limit (56), or by substituting  $C$  and  $a_i$  to the formulae (57). All these methods yield the same result:

$$y(x) = 1 + (a - 1)x + (a - a^2)x^2 - 2(a - 3a^2 + 2a^3)x^3 + \dots$$

Alternatively, starting from this result (obtained e.g. by solving  $A(x, y) = 0$ ), and comparing with coefficients in (57), we can reconstruct  $C$  and  $a_i$ .

Similarly, for the non-abelian branch of right-handed trefoil, the generating series (50) is determined by the quiver matrix (51),  $\mathbf{a} = (0, 1, 1, 0)$  (and appropriate  $\mathbf{q}$ ), so that

$$(59) \quad F(x, a, q) = 1 + x \frac{a}{q} + x^2 \frac{a(a - q)}{(q - 1)q} + \dots$$

Again we can determine the classical  $y(x)$  for this branch either by solving the  $A$ -polynomial equation  $A(x, y) = 0$ , or by taking the limit (56), or by substituting  $C$  and  $a_i$  to the formulae (57). All these methods yield the same result:

$$y(x) = 1 + (-2a + 2a^2)x^2 + (-3a^2 + 3a^3)x^3 + \dots$$

Alternatively, starting from this result (obtained e.g. by solving  $A(x, y) = 0$ ), and comparing with coefficients in (57), we can reconstruct  $C$  and  $a_i$ .

Analogously, the first (left-most) quiver matrix in (39), together with  $\mathbf{a} = (0, 0, 0, 1, 1, 1)$  (and appropriate  $\mathbf{l}$ ), determine the series (38) for the abelian branch of figure-eight knot:

$$F(x, a, q) = 1 + x \frac{3(a - q)}{q - 1} + x^2 \frac{a^2q^2 + 6a^2q + 2a^2 - aq^3 - 8aq^2 - 8aq - a + 2q^3 + 6q^2 + q}{(q - 1)^2(q + 1)} + \dots$$

Substituting  $C$  and  $a_i$  to (57) we also get

$$y(x) = 1 + 3(-1 + a)x + (1 - 9a + 8a^2)x^2 + (-1 + 6a - 27a^2 + 22a^3)x^3 + \dots$$

or we can reconstruct  $C$  and  $a_i$  by matching this classical series with coefficients in (57).

Finally, for the non-abelian branch (of slope  $-1/2$ ) of figure-eight knot, the matrix (41) and  $\mathbf{a} = (0, 1, 0, 0, 1)$  (and appropriate  $\mathbf{l}$ ) determine the series (40)

$$F(x, a, q) = 1 + x \frac{2q^2 - q}{q - 1} + x^2 \frac{-aq^4 + aq^3 + aq^2 - aq + q^6 + 3q^5 - 2q^4 - 2q^3 + q^2}{(q - 1)^2(q + 1)} + \dots$$

and the corresponding classical series reads

$$y(x) = 1 + x + 3x^2 + (9 - 3a)x^3 + \dots$$

4.3.2. *Extremal invariants and the reconstruction of non-abelian  $F_{\mathfrak{3}_1^r}$ .* Let us now illustrate in a detailed yet simple example how to determine  $F_K(x, a, q)$  from the classical function  $y(x)$ . Among others, interesting and sufficiently simple examples can be constructed in the extremal limit [GKS16], defined as follows. For a given  $F_K(x, a, q) = \sum_{n=0}^{\infty} f_n(a, q)x^n$  we take into account only the term with the lowest or the highest power of  $a$  in each  $f_n(a, q)$ , and ignore all other terms in  $f_n(a, q)$ . More specifically, we call such extremal invariants respectively as minimal or maximal. If  $F_K(x, a, q)$  is written in a quiver form (55), then the parameters  $a_i$  play a crucial role in determining its extremal version. The corresponding extremal  $A$ -polynomial (whose solution encodes the extremal  $y(x)$ ), can be also defined by taking an appropriate limit of the full  $A$ -polynomial [GKS16].

For concreteness, consider the non-abelian branch for the right-handed trefoil, whose  $F_K$  series is given in (50) and (59). Its quiver matrix (51) has size  $m = 4$  and  $\mathbf{a} = (0, 1, 1, 0)$ . The maximal invariants in this case are encoded in the submatrix of (51) given by  $C_{ij}$  for  $i, j = 2, 3$ . Indeed, since  $a_2 = a_3 = 1 > a_1 = a_4 = 0$ , the highest powers of  $a$  for a fixed  $x$  will arise from contributions from  $x_2$  and  $x_3$ . We thus end up with a non-trivial example with a quiver matrix of size  $m = 2$ , which still captures some interesting properties of trefoil knot. Let us relabel indices  $(2, 3)$  into  $(1, 2)$ , so that the extremal quiver generating series in this case takes form

$$(60) \quad F_{\mathfrak{3}_1^r}(x, a, q) = \sum_{d_1, d_2=0}^{\infty} (-q^{1/2})^{\binom{d_1}{\beta} \binom{d_2}{\gamma}} \frac{a^{a_1 d_1 + a_2 d_2} q^{q_1 d_1 + q_2 d_2} x^{d_1 + d_2}}{(q)_{d_1} (q)_{d_2}} =$$

$$= 1 + x \frac{a}{q} + x^2 \frac{a^2}{(q-1)q} + x^3 \frac{a^3 (q^2 + 1)}{(q-1)q^4} + x^4 \frac{a^4 (q^5 + q^2 - 1)}{(q-1)^2 q^6 (q+1)} + \dots,$$

with  $\binom{\alpha}{\beta} \binom{\beta}{\gamma} = \binom{0}{-1} \binom{-1}{1}$ ,  $a_1 = a_2 = 1$ , and  $(q_1, q_2) = (-1, 0)$ . The classical extremal series can be determined either from a limit of  $A$ -polynomial or as a limit

$$(61) \quad y(x) = \lim_{q \rightarrow 1} \frac{F_{\mathfrak{3}_1^r}(qx, a, q)}{F_{\mathfrak{3}_1^r}(x, a, q)} = 1 + 2a^2 x^2 + 3a^3 x^3 + 3a^4 x^4 + \dots$$

Suppose now that we know only the above classical series (e.g. from the analysis of classical  $A$ -polynomial). We reconstruct (60) as follows. First, consider coefficients of (61):

$$y_1 = 0, \quad y_2 = 2a^2, \quad y_3 = 3a^3, \quad y_4 = 3a^4,$$

and compare them to the general form (58). We observe that:

- We have  $y_1 = 0 \equiv -(-1)^\alpha a^{a_1} - (-1)^\gamma a^{a_2}$ , which means that  $\alpha$  and  $\gamma$  must be of the opposite parity and  $a_1 = a_2$  (so that the two terms indeed cancel).
- If we substitute  $a_1 = a_2$  into the equation for  $y_2$ , we get

$$y_2 = 2a^2 \equiv a^{2a_1} (\alpha + \gamma + (-1)^{\alpha+\gamma} (1 + 2\beta)) = a^{2a_1} (\alpha + \gamma - 1 - 2\beta),$$

where we took into account that  $(-1)^{\alpha+\gamma} = -1$  (because  $\alpha$  and  $\gamma$  have the opposite parity); we thus conclude that  $a_1 = a_2 = 1$  and  $\alpha + \gamma - 1 - 2\beta = 2$ .

- As  $\alpha$  and  $\gamma$  have the opposite parity, assume that  $\alpha$  is even and  $\gamma$  is odd (which can be justified a posteriori); then the expression for the third coefficient in (58) simplifies to

$$y_3 = 3a^3 \equiv -\frac{3}{2} a^3 (\alpha - \gamma) (\alpha + \gamma - 1 - 2\beta) = 3a^3 (\gamma - \alpha),$$

where in the last step we used  $\alpha + \gamma - 1 - 2\beta = 2$  determined above; this last equation implies that  $\gamma = \alpha + 1$ , and using this and  $\alpha + \gamma - 1 - 2\beta = 2$  we also get  $\beta = \alpha - 1$ .

- To summarize, so far we have found that  $a_1 = a_2 = 1$ ,  $\beta = \alpha - 1$ ,  $\gamma = \alpha + 1$ , and  $(-1)^{\alpha+\gamma} = -1$ ; substituting all this into (58), the formula for  $y_4$  reduces to

$$y_4 = 3a^4 \equiv (8\alpha + 3)a^4,$$

which therefore implies that  $\alpha = 0$ .

- Ultimately, we have found that

$$(62) \quad \alpha = 0, \quad \beta = \alpha - 1 = -1, \quad \gamma = \alpha + 1 = 1, \quad a_1 = a_2 = 1,$$

which fixes the quiver matrix and parameters  $a_i$  in (the first line of) (60).

So far we have taken advantage of the classical information. In addition, the quantum information that fully determines (60) is captured by  $l_i$ , which we can deduce by plugging the result (62) into the first line of (60) and comparing initial coefficients with the known result (assuming we can deduce it independently). In our example it turns out that the first non-trivial term in the expansion of (60), i.e.  $x \frac{a}{q}$ , fully fixes  $l_i$ . Indeed, plugging (62) into the first line of (60) we get

$$F_{\mathbf{3}_1^r}(x, a, q) = 1 + ax \frac{q^{l_2} - q^{l_1}}{q - 1} + \dots \equiv 1 + x \frac{a}{q} + \dots$$

The above comparison fixes  $l_1 = -1$  and  $l_2 = 0$  uniquely.

To sum up, from the analysis of  $y_2, y_3, y_4$  coefficients in the classical function  $y(x)$ , and then the first non-trivial term in  $F_{\mathbf{3}_1^r}(x, a, q)$ , we find the data

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \quad a_1 = a_2 = 1, \quad (l_1, l_2) = (-1, 0),$$

which fully determines the quiver generating series (60).

**4.4. Analogues of  $d$ -invariants and stabilization.** When we specialize to  $\mathfrak{sl}_2$ ,  $F_K$  has the form (ignoring prefactors)

$$\sum_{n=0}^{\infty} f_n(q) x^n.$$

Understanding the lowest power of  $q$  in  $f_n(q)$  – whose  $n^2$  coefficient is denoted by  $4c$  in [GM21]<sup>6</sup> – has several useful applications:

- It tells us when we can apply the surgery formula in its simplest form [GM21].
- It is an analogue of  $\Delta_a$  which appears as a prefactor for homological blocks [GPV17, GPPV20, GM21]:

$$\widehat{Z}(Y; q) \in q^{\Delta_a} \mathbb{Z}[[q]],$$

where  $a$  is a choice of  $\text{spin}^c$  structure on  $Y$ .

- It tells us by which power of  $q$  we should normalize  $f_n(q)$  so that the  $q$ -expansion starts with  $1 + (\dots)$ . This, in turn, is a key step in exploring stabilization of  $f_n(q)$  as  $n \rightarrow \infty$ .

---

<sup>6</sup> $c = \frac{1}{24}$  for  $\mathbf{3}_1^r$ ,  $c = -\frac{1}{24}$  for  $\mathbf{3}_1$ , and  $c = -\frac{1}{16}$  for  $\mathbf{4}_1$ .

It turns out that in certain cases it is possible to understand both the lowest power and stabilisation questions directly from a quiver. To provide a concrete example of this, we initially focus on the reduced  $F_K$  invariant for the figure eight knot.

We use the quiver from Equation (38), choosing the first of the 3 options for the  $C$  matrix and setting  $t = -1, a = q^2$ . The first step is to determine the growth of the lowest power of  $q$  which we call<sup>7</sup>  $\Delta$ . For large  $n$ , the  $q$  coefficient is dominated by the quadratic term so we want to solve the linear programming problem:<sup>8</sup>

$$(63) \quad \begin{aligned} & \text{minimise } \mathbf{d} \cdot C \cdot \mathbf{d} \\ & \text{subject to } 0 \leq \mathbf{d} \text{ and } \mathbf{n} \cdot \mathbf{d} = 1. \end{aligned}$$

Some solution spaces for simple knots are shown in Figure 4.

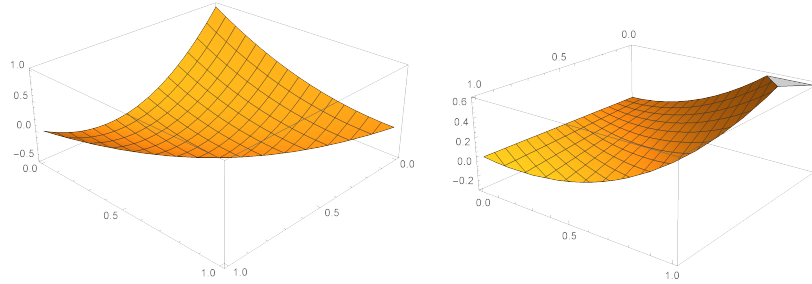


FIGURE 4. The value of the quadratic form in (63).

Left: the reduced  $4_1$  knot with  $(d_2, d_3, d_4) \in ([0, 1], [0, 1 - d_2], 1 - d_2 - d_3)$ .

Right: the unreduced  $3_1^r$  with  $(d_1, d_2, d_3) \in ([0, 1], [0, 1 - d_1], 1 - d_1 - d_2)$ .

We can often solve this by hand, making use of the observation that if we can find a pair  $(i, j)$  such that  $n_i = n_j$  and  $C_{ik} \leq C_{jk}$  for all  $k$ , then  $d_j = 0$ . The idea is simply that if such a pair exists, it is always advantageous to move weight from  $d_j$  to  $d_i$ .

In the case of the  $4_1$  knot only  $d_2$  and  $d_3$  survive and there is a unique solution

$$\mathbf{d} = \left( 0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0 \right).$$

This splits us into two cases depending on the parity of  $n$ . Let us start with the even parity case,  $n = 2i$ . Then for large  $i$ , the minimal  $q$  power of  $f_n(q)$  comes from the vector  $\mathbf{d} = (0, i, i, 0, 0, 0)$  and is given by

$$\Delta_{\text{even}} = \frac{1}{2} \mathbf{d} \cdot C \cdot \mathbf{d} + \mathbf{l} \cdot \mathbf{d} = 2i - i^2.$$

Note that the quadratic term  $-i^2$  matches the result from [GM21] once one accounts for the differences in notation. Next we want to determine the even stable series. The key question we need to answer is for what other vectors  $\mathbf{d}'$  satisfying  $\mathbf{n} \cdot (\mathbf{d} - \mathbf{d}') = 0$  will  $\frac{1}{2} \mathbf{d}' \cdot C \cdot \mathbf{d}' + \mathbf{l} \cdot \mathbf{d}'$  stay close to  $c_{\text{even}}$  as  $i \rightarrow \infty$ . This is solved by looking at the double derivative

$$\frac{d}{di} \left( \frac{d}{dd} \left( \frac{1}{2} \mathbf{d} \cdot C \cdot \mathbf{d} + \mathbf{l} \cdot \mathbf{d} \right) \Big|_{\mathbf{d}=(0,i,i,0,0,0)} \right) = (0, -1, -1, -1, 1, 0).$$

<sup>7</sup>This is closely related to  $c$ , but  $c$  only looks at the quadratic growth.

<sup>8</sup> $\mathbf{d}$  represents  $\frac{\mathbf{d}}{n}$  from the quiver form.

This means that as long as the sum  $d_2 + d_3 + d_4$  stays constant,  $\frac{1}{2}\mathbf{d}' \cdot C \cdot \mathbf{d}' + \mathbf{l} \cdot \mathbf{d}'$  will not diverge as we increase  $i$ . Hence we consider vectors of the form  $(0, i - l, i - m + l, m, 0, 0)$  with  $m \geq 0$  and  $|l|, |m| \ll i$ . The quiver term corresponding to a vector of this form is

$$q^{2i-i^2} \frac{(-1)^m q^{l^2 + \frac{1}{2}(m^2+m)}}{(q)_{i-l}(q)_{i-m+l}(q)_m} x^i.$$

Taking the limit as  $i$  goes to infinity and summing over  $m$  and  $l$  gives

$$\begin{aligned} \text{Stable}_{\text{even}}(q) &= \frac{1}{(q)_\infty^2} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{1}{2}(m^2+m)}}{(q)_m} \sum_{l=-\infty}^{\infty} q^{l^2} \\ &= \frac{1 + 2q + 2q^4 + 2q^9 + \dots}{(q)_\infty} \\ &= 1 + 3q + 4q^2 + 7q^3 + 13q^4 + 19q^5 + 29q^6 + 43q^7 + 62q^8 + 90q^9 + \dots \end{aligned}$$

This can be recognized as the ratio of Ramanujan theta functions  $\psi(q)/f(-q)$  or, equivalently, as a  $q$ -series expansion of  $q^{1/24} \frac{\eta(q^2)^5}{\eta(q)^3 \eta(q^4)^2}$ .

For the odd case  $j = 2i + 1$ , the process is identical. The optimal vector is either<sup>9</sup>  $(0, i + 1, i, 0, 0, 0)$  or  $(0, i, i + 1, 0, 0, 0)$ , which both lead to  $c_{\text{odd}} = 1 + i - i^2$  and give stable series

$$\begin{aligned} \text{Stable}_{\text{odd}}(q) &= \frac{1}{(q)_\infty^2} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{1}{2}(m^2+m)}}{(q)_m} \sum_{l=-\infty}^{\infty} q^{l^2-l} \\ &= 2 \frac{1 + q^2 + q^6 + q^{10} + \dots}{(q)_\infty} \\ &= 2(1 + q + 3q^2 + 4q^3 + 7q^4 + 10q^5 + 17q^6 + 23q^7 + 35q^8 + 48q^9 + \dots). \end{aligned}$$

This series can be recognized as a ratio of Ramanujan theta functions,  $\psi(q^2)/f(-q)$  or, equivalently, as a  $q$ -series expansion of  $q^{-5/24} \frac{\eta(q^4)^2}{\eta(q)\eta(q^2)}$ . It also has a nice enumerative interpretation, as the generating function of the number of partitions of  $n$  in which each odd part can occur any number of times but each even part is of two kinds and each kind can occur at most once:

$$\frac{1}{(q; q^2)_\infty (q^2; q^4)_\infty} = \prod_{k=1}^{\infty} (1 + q^k) (1 + q^{2k})^2.$$

Finally, observe that as both  $\Delta_{\text{even}}$  and  $\Delta_{\text{odd}}$  are quadratic in  $i$ , passing to the unreduced case – which, up to an overall prefactor, corresponds to multiplying by  $(1 - qx)$  – will have no effect on the stable series and simply apply a small shift to the minimal  $q$  power. The reason we make this comment is that in some cases, most notably that of  $(2, 2p + 1)$  torus knots, passing to the unreduced case leads to a massive simplification in coefficients. For example for the right-handed trefoil

$$\begin{aligned} F_{\mathfrak{3}_1}(x, q) &= 1 + \frac{x}{q} + \frac{1 - q}{q^2} x^2 + \frac{1 - q - q^2}{q^3} x^3 + \frac{1 - q - q^2}{q^4} x^4 + \frac{1 - q - q^2 + q^5}{q^4} x^5 + \dots \\ F_{\mathfrak{3}_1}^{\text{unreduced}}(x, q) &= 1 - \frac{x^2}{q} - \frac{x^3}{q} + x^5 + qx^6 - q^4 x^8 + \dots \end{aligned}$$

<sup>9</sup>In general, there is usually a difference between between different symmetry breaking choices. See (65)

We will return to this example later.

For most purposes, understanding  $\Delta$  and the stable series for the unreduced case is more useful so we will focus on that case going forward. One immediate issue this presents is that the quivers we have constructed so far have been all for the reduced case, as it is not immediately apparent how to convert between an unreduced and reduced quiver form. The key observation is that instead of passing to the full quiver, we can work with an intermediate expression where

$$F_K^{\text{reduced}}(x, q) = \sum_{k, \dots} (\dots)(x, q^{-1})_k,$$

and then the unreduced invariant is simply

$$F_K^{\text{unreduced}}(x, q) = \sum_{k, \dots} (\dots)(qx, q^{-1})_{k+1},$$

and from here we can apply (33). Observe that when we apply this method, the  $C$  matrix of the reduced and unreduced quivers are identical and only the linear terms change.

Let us study the left-handed trefoil next. Taking the  $a \rightarrow q^2$  limit of the formula in [EGG<sup>+</sup>20], we find that the unreduced  $\mathfrak{sl}_2$  invariant is

$$(64) \quad F_{\mathfrak{sl}_2}^{\text{unreduced}}(x, q) = \sum_{k=0}^{\infty} (xq)^k (qx, q^{-1})_{k+1} = \sum_{i, j, k=0}^{\infty} \frac{(-1)^i x^{i+j+k} q^{i+2j+k-ki+\frac{1}{2}(i^2-i)}}{(q)_i (q)_j}.$$

While this is not technically a quiver form, as there is no  $(q)_k$  in the denominator, this is good enough for our purposes. It can be made into a quiver form by inserting 1 and using the general identity

$$\sum_{\alpha+\beta=k} (-1)^i \frac{q^{\frac{\alpha^2+\alpha}{2}}}{(q)_\alpha (q)_\beta} = 1,$$

which gives the unreduced version of the quiver in (43). Since this does not change any result, we will continue with the pseudo-quiver form of (64), which can be written as

$$F_{\mathfrak{sl}_2}^{\text{unreduced}}(x, q) = \sum_{\mathbf{d}} \frac{(-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d} + \mathbf{l} \cdot \mathbf{d}}}{(q)_{d_2} (q)_{d_3}}, \quad \mathbf{C} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{matrix} \mathbf{n} = (1, 1, 1), \\ \mathbf{l} = (1, \frac{1}{2}, 2). \end{matrix}$$

From here we analyse in an identical manner. Minimizing the quadratic term we find that there is a unique solution  $(\frac{2}{3}, \frac{1}{3}, 0)$ , so we split the analysis of  $x^j$  into three cases depending on  $j \pmod 3$  with the following  $\Delta$  values:

$$(65) \quad \begin{cases} j = 3i, & \mathbf{d} = (2i, i, 0), & \Delta = \frac{5}{2}i - \frac{3}{2}i^2, \\ j = 3i + 1, & \mathbf{d} = (2i + 1, i, 0), & \Delta = 1 + \frac{3}{2}i - \frac{3}{2}i^2, \\ j = 3i + 2, & \mathbf{d} = (2i + 1, i + 1, 0), & \Delta = 1 + \frac{1}{2}i - \frac{3}{2}i^2. \end{cases}$$

Note that the quadratic coefficient  $-\frac{3}{2}$  agrees with  $c = -\frac{1}{24}$  in [GM21] after aligning notations. Next we compute

$$\frac{d}{di} \left( \frac{d}{d\mathbf{d}} \left( \frac{1}{2} \mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d} + \mathbf{l} \cdot \mathbf{d} \right) \Big|_{\mathbf{d}=(2i, i)} \right) = (-1, -1, 0),$$

and so the three stable series are

$$\begin{cases} j = 3i, & \mathbf{d} = (2i, i, 0), & \frac{(-1)^i}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2}{2} + \frac{k}{2}} = (-1)^i, \\ j = 3i + 1, & \mathbf{d} = (2i + 1, i, 0), & \frac{(-1)^i}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2}{2} + \frac{3k}{2}} = 0, \\ j = 3i + 2, & \mathbf{d} = (2i + 1, i + 1, 0), & \frac{(-1)^{i+1}}{(q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2}{2} - \frac{k}{2}} = (-1)^{i+1}. \end{cases}$$

The summation in the middle is 0 as  $k^2 + k = (-1 - k)^2 + (-1 - k)$ . Note that this does not mean that the coefficients of  $x^{3i+1}$  are 0. All we can say is that the minimal  $q$  power of  $x^{3i+1}$  grows faster than  $-\frac{3}{2}i^2$ . Skipping over this for now, we find the expected structure for large  $i$  both in terms of  $q$  powers and the  $\pm 1$  stable series.

It is worth noting that this procedure is highly sensitive to the initial choice of quiver. For an example of what can go wrong if one does not start with a ‘‘good’’ quiver form, let us study the right-handed trefoil. We can generate a quiver form using (64), sending  $x \rightarrow x^{-1}$  and  $q \rightarrow q^{-1}$ , and then using Weyl symmetry to send  $x^{-1} \rightarrow q^2 x$ . From this we get

$$F_{3_1}^{\text{unreduced}}(x, q) = \sum_{k=0}^{\infty} (xq)^k (qx, q)_{k+1} = \sum_{i,j,k=0}^{\infty} \frac{(-1)^i x^{i+j+k} q^{i+2j+k+kj+\frac{1}{2}(i^2-i)}}{(q)_i (q)_j},$$

so the pseudo-quiver data (with  $d_1 = k$ ,  $d_2 = j$ ,  $d_3 = i$ ) reads

$$F_{3_1}^{\text{unreduced}}(x, q) = \sum_{\mathbf{d}} \frac{(-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} q^{\mathbf{l} \cdot \mathbf{d}} x^{\mathbf{n} \cdot \mathbf{d}}}{(q)_{d_2} (q)_{d_3}}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{matrix} \mathbf{n} = (1, 1, 1), \\ \mathbf{l} = (1, 2, \frac{1}{2}). \end{matrix}$$

In this case, there are two minima for the quadratic term,  $\mathbf{d} = (1, 0, 0)$  and  $\mathbf{d} = (0, 1, 0)$ . We differentiate these by looking at the linear terms. As  $\mathbf{l}_1 < \mathbf{l}_2$ , we conclude that the minimal  $q$  power comes from  $\mathbf{d} = (1, 0, 0)$ . Computing the derivative

$$\frac{d}{di} \left( \frac{d}{d\mathbf{d}} \left( \frac{1}{2} \mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d} + \mathbf{l} \cdot \mathbf{d} \right) \Big|_{\mathbf{d}=(i,0,0)} \right) = (0, 1, 0),$$

we see that the stable series will involve summing over  $(i - k, 0, k)$  for  $k \geq 0$ . Substituting this in, we find that the corresponding quiver term is

$$(-1)^k q^{i + \frac{k^2}{2} - \frac{k}{2}},$$

therefore the stable series reads

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{k^2}{2} - \frac{k}{2}}}{(q)_k} = (1, q)_\infty = 0.$$

This is to be expected, as for large  $i$  it is known that the  $q$  power grows quadratically. It is easy to check that expanding around the second smallest minimum,  $\mathbf{d} = (0, 1, 0)$ , the stable series are again 0. Hence we cannot get an accurate picture of either the stable series or large growth in  $q$  from this quiver.

It is an interesting question how to find ‘‘good’’ quiver forms in general.

5. QUIVERS FROM  $R$ -MATRICES AND INVERTED HABIRO SERIES

In this section we explain how to use  $R$ -matrices to get quiver expressions of  $F_K$  for  $\mathfrak{sl}_2$  (or even symmetrically coloured  $\mathfrak{sl}_N$ ) for a large class of knots. We also explain how to get quiver expressions from the inverted Habiro series in case of the trefoil knot and the figure-eight knot. In a few explicit examples, we observe that these quiver expressions can be lifted to the  $a$ -deformed version. This provides a yet another method to obtain the knot complement quivers for the abelian branch.

**5.1. Quivers from  $R$ -matrices:  $\mathfrak{sl}_2$  case.** The *large-colour  $R$ -matrix* studied in [Par20b, Par21] is the large-colour limit of the  $R$ -matrix for the  $n$ -coloured Jones polynomials. Explicitly, they are given by<sup>10</sup>

$$(66) \quad \check{R}(x)_{i,j}^{i',j'} = \delta_{i+j,i'+j'} q^{\frac{j+j'+1}{2}} x^{-\frac{i+j'+1}{2}} q^{jj'} \begin{bmatrix} i \\ j' \end{bmatrix}_{q^{-1}} \prod_{1 \leq l \leq i-j'} (1 - q^{j+l} x^{-1}),$$

$$\check{R}^{-1}(x)_{i,j}^{i',j'} = \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{i+i'+1}{2}} q^{-ii'} \begin{bmatrix} j \\ i' \end{bmatrix}_{q^{-1}} \prod_{1 \leq l \leq j-i'} (1 - q^{-i-l} x).$$

The  $R$ -matrix and its inverse correspond to the positive and the negative crossing as in the figure below. Throughout this section, as in Figure 5, we use the indices  $i, j, i', j'$  to denote

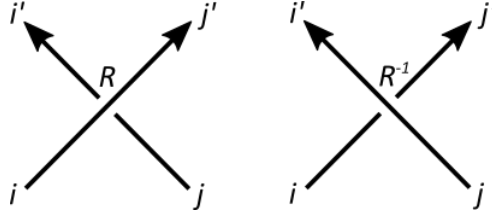


FIGURE 5. Positive and negative crossings

the segment on the bottom left, bottom right, top left, and the top right of the crossing, respectively.

Since we can expand  $q$ -Pochhammer symbols using the  $q$ -binomial theorem, it is easy to see that the  $R$ -matrix can be written as a sum of (monomial times)  $q$ -multinomial coefficients. This observation is very useful in finding a quiver expression for  $F_K$ . In fact, we have the following theorem:

**Theorem 1.** *For any positive braid knot  $K$ , there is an algorithm to produce a quiver form of  $F_K(x, q)$  from the  $R$ -matrix state sum.*

*Proof.* We assume that the reader is familiar with the  $R$ -matrix state sum described in [Par20b]. We set the weight of the open strand to be 0. Let  $c$  be the number of crossings. The number of internal segments is  $\frac{4c-2}{2} = 2c - 1$ . The number of conditions ' $i + j = i' + j'$ ' is  $c$ , but only  $c - 1$  of them are independent. Therefore, there are (at most)  $c$  free parameters in choosing the weights of a state. Our first step is to find a nice set of free parameters.

Let us focus on the segments that correspond to ' $j'$ ' for some crossing (i.e. the top right segment of a crossing). Let  $X$  be the set of such segments. It is in one-to-one correspondence

<sup>10</sup>Here,  $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}$  where  $[n]_q := \frac{1-q^{n+1}}{1-q}$ .

with the set of crossings. Obviously,  $|X| = c$  and  $X$  is naturally partitioned into those that are connected to form a diagonal over-strand (chain of segments) from bottom left to top right. Let us write  $X = \sqcup_{\alpha \in I} X_\alpha$  with some index set  $I$ . In each group  $\alpha$ , let us say  $X_\alpha = \{s_1^\alpha, s_2^\alpha, \dots, s_{l_\alpha}^\alpha\}$  where  $s_1^\alpha$  is the top right segment,  $s_2^\alpha$  is the one right below it, etc., and  $l_\alpha$  is the length of the chain of segments in  $X_\alpha$ ; That is,  $s_2^\alpha$  is ‘ $i$ ’ of a crossing for which  $s_1^\alpha$  is ‘ $j$ ’, and so on. Let  $n_{s_i^\alpha}$  denote the weight associated to the segment  $s_i^\alpha$ . (By our definition, each  $n_{s_i^\alpha}$  is ‘ $j$ ’ for some crossing.) Clearly  $0 \leq n_{s_1^\alpha} \leq n_{s_2^\alpha} \leq \dots \leq n_{s_{l_\alpha}^\alpha}$ . The state sum can be expressed as a summation over  $n_s$  for all  $s \in X$  (or more precisely, for all  $s$  except those whose has to be 0 because of the weight 0 on the open strand). This is because all the right-most segments are in  $X$ , and from there, column by column, all the other weights are uniquely fixed. Let  $m_{s_k^\alpha} = n_{s_k^\alpha} - n_{s_{k-1}^\alpha}$  for  $k = 2, \dots, l_\alpha$  and  $m_{s_1^\alpha} = n_{s_1^\alpha}$ . (See Figure 6 for an illustration.) With this change of variables, the state sum can now be expressed as

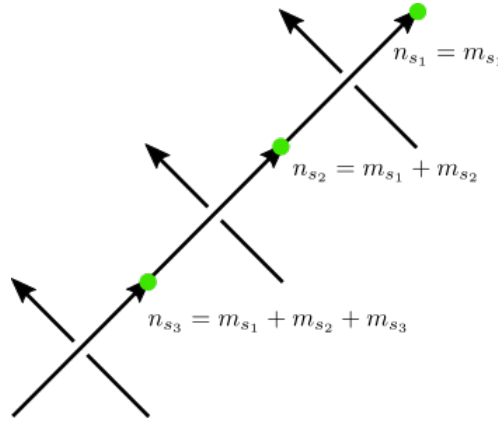


FIGURE 6. The preferred parameters  $m_{s_k}$  for  $\alpha \in I$

a summation over  $m_s$  with  $m_s \geq 0$ . It is easy to see that all the internal weights which are either ‘ $j$ ’ or ‘ $j$ ’ for some crossing, are  $\mathbb{N}$ -linear combinations of  $\{m_s\}_{s \in X}$ . In particular, in the state sum the degree of  $x$  is bounded above by  $-\sum_{s \in X} a_s m_s$  for some  $a_s > 0$ , which in particular implies the convergence of the state sum.

With this description of the state sum at hand, it is easy to turn it into a quiver form. Recall that

$$\begin{aligned} \check{R}(x)_{i,j}^{i',j'} &= \delta_{i+j,i'+j'} q^{\frac{j+j'+1}{2}} x^{-\frac{j+j'+1}{2}} q^{jj'} \begin{bmatrix} i \\ j' \end{bmatrix}_q \sum_{k=0}^{i-j'} q^{\frac{k(k+1)}{2}} \begin{bmatrix} i-j' \\ k \end{bmatrix}_q (-q^j x^{-1})^k \\ &= \delta_{i+j,i'+j'} q^{\frac{j+j'+1}{2}} x^{-\frac{j+j'+1}{2}} q^{jj'} \begin{bmatrix} i \\ j' \end{bmatrix}_q \sum_{k \geq 0} (-1)^k q^{\frac{k(k+1)}{2} + jk} x^{-k} \frac{(q^{i-j'-k+1})_k}{(q)_k}. \end{aligned}$$

The  $q$ -binomial coefficient involving  $k$  is already in a good form where we can use Lemma 4.5 of [KRSS19], so let us focus on the other  $q$ -binomial coefficients,  $\begin{bmatrix} i \\ j' \end{bmatrix}_q$ . For each  $\alpha \in I$ , the product of the  $q$ -binomial coefficients looks like

$$\begin{bmatrix} m_{s_1^\alpha} + m_{s_2^\alpha} \\ m_{s_1^\alpha} \end{bmatrix}_q \begin{bmatrix} m_{s_1^\alpha} + m_{s_2^\alpha} + m_{s_3^\alpha} \\ m_{s_1^\alpha} + m_{s_2^\alpha} \end{bmatrix}_q \dots \begin{bmatrix} i \\ m_{s_1^\alpha} + \dots + m_{s_{l_\alpha}^\alpha} \end{bmatrix}_q,$$

where  $i$  denotes the ‘ $i$ ’ of the crossing whose ‘ $j'$ ’ is  $n_{s_{l_\alpha}^\alpha}$ . This product of  $q$ -binomials can be simplified as

$$\frac{(q^{i-m_{s_1}-\dots-m_{s_l}+1})_{m_{s_1}+\dots+m_{s_l}}}{(q)_{m_{s_1}} \cdots (q)_{m_{s_l}}},$$

where we have dropped  $\alpha$  for simplicity of notation. In this form, the application of the Lemma 4.5 finishes the task. All in all, we get a quiver form of  $F_K(x, q)$  with at most  $4c$  nodes. In practice, it will be smaller than that, because there can be many segments whose weight has to be 0 because of the weight 0 on the open strand.  $\square$

This theorem can be extended to homogeneous braid knots as well, using the inverted state sum [Par21].

**Theorem 2.** *For any homogeneous braid knot  $K$ , there is an algorithm to produce a quiver form of  $F_K(x, q)$  from the inverted state sum.*

*Proof.* We proceed similarly to the proof of the previous theorem. Given a homogeneous braid, we focus on the segments that correspond to ‘ $j'$ ’ for some positive crossing and the segments that correspond to ‘ $j$ ’ for some negative crossing. Let  $X$  be the set of such segments. Again,  $X$  is naturally partitioned into those that are connected to form a diagonal over-strand. Let us write  $X = \bigsqcup_{\alpha \in I} X_\alpha \sqcup \bigsqcup_{\beta \in J} X_\beta$ , where  $I$  is an index set for the diagonal over-strands from bottom left to top right, and  $J$  is an index set for the diagonal over-strands from bottom right to top left. If we use the same notation for the weights as before, with  $s_1$  denoting the right-most segment,  $s_2$  the second right-most segment and so on,

$$0 \leq n_{s_1^\alpha} \leq \dots \leq n_{s_{l_\alpha}^\alpha} \quad \text{for any } \alpha \in I \quad \text{and} \quad 0 > n_{s_1^\beta} \geq \dots \geq n_{s_{l_\beta}^\beta} \quad \text{for any } \beta \in J.$$

Dividing the braid along vertical lines, we see by a simple ‘conservation of weight’ argument that the overall  $x^{-1}$ -degree of a given configuration is (ignoring the constant part)

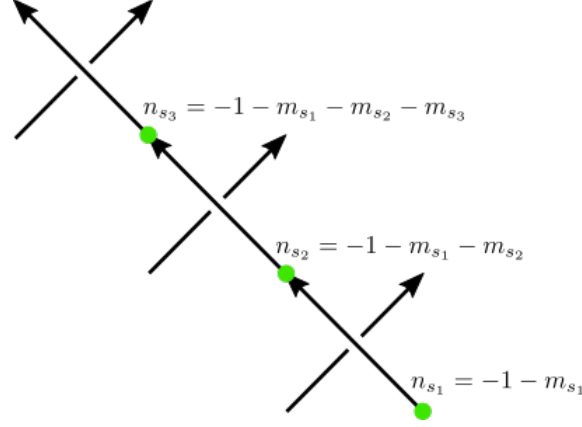
$$\sum_{\alpha \in I} \sum_{1 \leq k \leq l_\alpha} n_{s_k^\alpha} - \sum_{\beta \in J} \sum_{1 \leq k \leq l_\beta} n_{s_k^\beta}.$$

This shows first that the inverted state sum converges absolutely, and also that we have a set of elementary generators

$$m_{s_k^\alpha} = n_{s_k^\alpha} - n_{s_{k-1}^\alpha}, \quad \alpha \in I \quad \text{and} \quad m_{s_k^\beta} = -(n_{s_k^\beta} - n_{s_{k-1}^\beta}), \quad \beta \in J,$$

where, by convention, we set  $n_{s_0^\alpha} = 0$  and  $n_{s_0^\beta} = -1$  (see Figure 7 for an illustration). Now recall that

$$\check{R}(x)_{i,j}^{i',j'} = \begin{cases} \delta_{i+j,i'+j'} q^{\frac{j+j'+1}{2}} x^{-\frac{j+j'+1}{2}} q^{jj'} \begin{bmatrix} i \\ i-j' \end{bmatrix}_q \prod_{1 \leq l \leq i-j'} (1 - q^{j+l} x^{-1}) & \text{if } i \geq j' \geq 0 \\ & \text{or } 0 > i \geq j', \\ \delta_{i+j,i'+j'} q^{\frac{j+j'+1}{2}} x^{-\frac{j+j'+1}{2}} q^{jj'} \begin{bmatrix} i \\ j' \end{bmatrix}_q \frac{1}{\prod_{0 \leq l \leq j'-i-1} (1 - q^{j-l} x^{-1})} & \text{if } j' \geq 0 > i, \\ 0 & \text{otherwise,} \end{cases}$$

FIGURE 7. The preferred parameters  $m_{s_k}$  for  $\beta \in J$ 

and

$$\check{R}^{-1}(x)_{i,j}^{i',j'} = \begin{cases} \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{i+i'+1}{2}} q^{-ii'} \begin{bmatrix} j \\ j-i' \end{bmatrix}_{q^{-1}} \prod_{1 \leq l \leq j-i'} (1 - q^{-i-l}x) & \text{if } j \geq i' \geq 0 \\ & \text{or } 0 > j \geq i', \\ \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{i+i'+1}{2}} q^{-ii'} \begin{bmatrix} j \\ i' \end{bmatrix}_{q^{-1}} \frac{1}{\prod_{0 \leq l \leq i'-j-1} (1 - q^{-i+l}x)} & \text{if } i' \geq 0 > j, \\ 0 & \text{otherwise.} \end{cases}$$

We have already analyzed positive crossings with  $i \geq j' \geq 0$ , so let us take a look at  $j' \geq 0 > i$ . In that case

$$\begin{aligned} \check{R}(x)_{i,j}^{i',j'} &= \delta_{i+j,i'+j'} q^{\frac{j+j'+1}{2}} x^{-\frac{j+j'+1}{2}} q^{jj'} \begin{bmatrix} i \\ j' \end{bmatrix}_q \sum_{k \geq 0} \begin{bmatrix} j' - i + k - 1 \\ k \end{bmatrix}_{q^{-1}} (q^j x^{-1})^k \\ &= \delta_{i+j,i'+j'} q^{\frac{j+j'+1}{2}} x^{-\frac{j+j'+1}{2}} q^{jj'} \begin{bmatrix} i \\ j' \end{bmatrix}_q \sum_{k \geq 0} q^{(i'+1)k} x^{-k} \frac{(q^{j'-i})_k}{(q)_k}. \end{aligned}$$

The  $q$ -binomial coefficient involving  $k$  is already in a good form where we can use Lemma 4.5.

The other  $q$ -binomial coefficient  $\begin{bmatrix} i \\ j' \end{bmatrix}_q$  can be handled in the same way we did in the proof of the previous theorem.

Now that we are done with positive crossings, let us take a look at negative crossings. When  $0 > j \geq i'$ , we have

$$\begin{aligned} \check{R}^{-1}(x)_{i,j}^{i',j'} &= \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{i+i'+1}{2}} q^{-i'j'} \begin{bmatrix} j \\ j-i' \end{bmatrix}_q \sum_{k=0}^{j-i'} q^{-\frac{k(k+1)}{2}} \begin{bmatrix} j-i' \\ k \end{bmatrix}_{q^{-1}} (-q^{-i}x)^k \\ &= \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{i+i'+1}{2}} q^{-i'j'} \begin{bmatrix} j \\ j-i' \end{bmatrix}_q \sum_{k=0}^{j-i'} q^{-\frac{(j-i'-k)(j-i'-k+1)}{2}} \begin{bmatrix} j-i' \\ k \end{bmatrix}_{q^{-1}} (-q^{-i}x)^{j-i'-k} \\ &= \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{j+j'+1}{2}} q^{-i'j'} \begin{bmatrix} j \\ j-i' \end{bmatrix}_q \sum_{k \geq 0} (-1)^{j-i'-k} q^{-\frac{(j-i'-k)(i+j'+k+1)}{2}} x^{-k} \frac{(q^{j-i'-k+1})_k}{(q)_k}. \end{aligned}$$

The  $q$ -binomial coefficient involving  $k$  is already in a good form, so we can focus on the other one,  $\left[ \begin{smallmatrix} j \\ j-i' \end{smallmatrix} \right]_q$ . For each  $\beta \in J$ , the product of the  $q$ -binomial coefficients look like

$$\left[ \begin{smallmatrix} -1 - m_{s_1^\beta} \\ m_{s_2^\beta} \end{smallmatrix} \right]_q \left[ \begin{smallmatrix} -1 - m_{s_1^\beta} - m_{s_2^\beta} \\ m_{s_3^\beta} \end{smallmatrix} \right]_q \cdots \left[ \begin{smallmatrix} -1 - m_{s_1^\beta} - \cdots - m_{s_{l_\beta}^\beta} \\ i' \end{smallmatrix} \right]_q,$$

where  $i'$  denotes the ' $i'$ ' of the crossing whose ' $j$ ' is  $n_{s_{l_\beta}^\beta}$ . Using the fact that

$$\left[ \begin{smallmatrix} -1 - n \\ k \end{smallmatrix} \right]_q = (-1)^k q^{-nk - \frac{k(k+1)}{2}} \left[ \begin{smallmatrix} n+k \\ n \end{smallmatrix} \right]_q,$$

we see that the product of  $q$ -binomials can be simplified as (up to an overall sign and  $q$  power)

$$\left[ \begin{smallmatrix} m_{s_1} + m_{s_2} \\ m_{s_1} \end{smallmatrix} \right]_q \left[ \begin{smallmatrix} m_{s_1} + m_{s_2} + m_{s_3} \\ m_{s_1} + m_{s_2} \end{smallmatrix} \right]_q \cdots \left[ \begin{smallmatrix} m_{s_1} + \cdots + m_{s_l} + i' \\ m_{s_1} + \cdots + m_{s_l} \end{smallmatrix} \right]_q = \frac{(q^{i'+1})_{m_{s_1} + \cdots + m_{s_l}}}{(q)_{m_{s_1}} \cdots (q)_{m_{s_l}}},$$

where we have dropped  $\beta$  for simplicity of notation. In this form we can apply Lemma 4.5.

The last case we need to consider is the negative crossing with  $i' \geq 0 > j$ , for which we have

$$\begin{aligned} \check{R}^{-1}(x)_{i,j}^{i',j'} &= \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{i+i'+1}{2}} q^{-i'j'} \left[ \begin{smallmatrix} j \\ i' \end{smallmatrix} \right]_q \frac{(-1)^{i'-j} q^{\frac{(i'-j)(i+j'+1)}{2}} x^{-(i'-j)}}{\prod_{0 \leq l \leq i'-j-1} (1 - q^{i-l} x^{-1})} \\ &= \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{j+j'+1}{2}} q^{-i'j'} \left[ \begin{smallmatrix} j \\ i' \end{smallmatrix} \right]_q \sum_{k \geq 0} (-1)^{i'-j} q^{\frac{(i'-j)(i+j'+1)}{2}} \left[ \begin{smallmatrix} i'-j+k-1 \\ k \end{smallmatrix} \right]_{q^{-1}} (-q^i x^{-1})^k \\ &= \delta_{i+j,i'+j'} q^{-\frac{i+i'+1}{2}} x^{\frac{j+j'+1}{2}} q^{-i'j'} \left[ \begin{smallmatrix} j \\ i' \end{smallmatrix} \right]_q \sum_{k \geq 0} (-1)^{i'-j+k} q^{\frac{(i'-j)(i+j'+1)}{2} + k(j'+1)} x^{-k} \frac{(q^{i'-j})_k}{(q)_k}. \end{aligned}$$

The  $q$ -binomial coefficient involving  $k$  is in a good form where Lemma 4.5 can be applied.

The other  $q$ -binomial coefficient  $\left[ \begin{smallmatrix} j \\ i' \end{smallmatrix} \right]_q$  was already considered above.

All in all, we have shown that for any homogeneous braid knot  $K$ , there is a quiver form of  $F_K(x, q)$ , with the size of the quiver being at most  $4c$ .  $\square$

**5.1.1. Right-handed trefoil.** Let us take a look at the right-handed trefoil knot as our first example in this section. Take the braid  $\sigma_1^3$ . The reduced  $F_{\mathfrak{B}_1^3}(x, q)$  can be expressed as a power series in  $x^{-1}$  in the following way:<sup>11</sup>

$$\begin{aligned} F_{\mathfrak{B}_1^3}(x, q) &= \sum_{m \geq 0} x^{\frac{1}{2}} q^{-\frac{1}{2}-m} \check{R}(x)_{0,m}^{m,0} \check{R}(x)_{m,0}^{m,0} \check{R}(x)_{m,0}^{0,m} \\ &= x^{-1} q \sum_{m \geq 0} \sum_{k=0}^m (-1)^k x^{-m-k} q^{\frac{k(k+1)}{2}} \frac{(q)_m}{(q)_k (q)_{m-k}} \\ &= x^{-1} q \sum_{d_1, d_2 \geq 0} (-1)^{d_1} x^{-2d_1-d_2} q^{\frac{d_1(d_1+1)}{2}} \frac{(q)_{d_1+d_2}}{(q)_{d_1} (q)_{d_2}}. \end{aligned}$$

<sup>11</sup>If one wants to work with power series in  $x$ , one simply needs to replace  $x^{-1}$  by  $x$ , thanks to Weyl symmetry.

Using Lemma 4.5 again, we can write

$$\begin{aligned}
F_{\mathbf{3}_1^r}(x, q) &= x^{-1}q \sum_{\alpha_1, \beta_1, \alpha_2, \beta_2 \geq 0} (-1)^{\alpha_1 + \beta_1} x^{-2(\alpha_1 + \beta_1) - (\alpha_2 + \beta_2)} q^{\frac{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)}{2}} \\
&\quad \times \frac{(-1)^{\alpha_1 + \alpha_2} q^{\frac{\alpha_1(\alpha_1 + 1)}{2} + \frac{\alpha_2(\alpha_2 + 1)}{2} + \alpha_2(\alpha_1 + \beta_1)}}{(q)_{\alpha_1} (q)_{\beta_1} (q)_{\alpha_2} (q)_{\beta_2}} \\
&= x^{-1}q \sum_{\mathbf{d}} (-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} q^{\mathbf{l}^{\text{sl}_2} \cdot \mathbf{d}}}{(q)_{\mathbf{d}}},
\end{aligned}$$

where

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{n} = (-2, -2, -1, -1), \quad \mathbf{l}^{\text{sl}_2} = \left(1, \frac{1}{2}, \frac{1}{2}, 0\right).$$

Note that we could have used Lemma 4.5 in a slightly different way, in which case we would have  $\alpha_1(\alpha_2 + \beta_2)$  instead of  $\alpha_2(\alpha_1 + \beta_1)$ , and the corresponding quiver matrix would be

$$\mathbf{C}' = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In general, there are many ways to apply the Lemma 4.5 which lead to different but equivalent quivers – for a thorough study of this phenomenon see [JKL<sup>+</sup>21].

Looking at the first few coefficients, we can easily upgrade this to the  $a$ -deformed version. Up to a prefactor, we have

$$F_{\mathbf{3}_1^r}(x, a, q) = \sum_{\mathbf{d}} (-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} a^{\mathbf{a} \cdot \mathbf{d}} q^{\mathbf{l} \cdot \mathbf{d}}}{(q)_{\mathbf{d}}},$$

where

$$\mathbf{n} = (2, 2, 1, 1), \quad \mathbf{a} = (2, 1, 1, 0), \quad \mathbf{l} = \left(-3, -\frac{3}{2}, -\frac{3}{2}, 0\right).$$

5.1.2. *Figure-eight knot.* As our next example, let us take a look at the figure-eight knot, given as the closure of the braid  $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ . In terms of the power series in  $x^{-1}$  we have

$$\begin{aligned}
F_{4_1}(x, q) &= - \sum_{\substack{m \geq 0 \\ k < 0}} xq^{-1-m-k} \check{R}(x)_{0,m}^{m,0} \check{R}(x)_{m,0}^{0,m} \check{R}^{-1}(x)_{0,k}^{0,k} \check{R}^{-1}(x)_{m,k}^{m,k} \\
&= -x^{-1} \sum_{m,k,a,b \geq 0} x^{-m-2k-a-b} q^{-k^2-ak-bk} \frac{(q)_{m+k+b}}{(q)_m(q)_k(q)_b} \frac{(q)_{k+a}}{(q)_k(q)_a} \\
&\stackrel{\text{lemma 4.5}}{=} -x^{-1} \sum_{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4 \geq 0} x^{-(\alpha_1+\beta_1)-2(\alpha_2+\beta_2)-(\alpha_4+\beta_4)-(\alpha_3+\beta_3)} \\
&\quad \times q^{-(\alpha_2+\beta_2)^2-(\alpha_4+\beta_4)(\alpha_2+\beta_2)-(\alpha_3+\beta_3)(\alpha_2+\beta_2)} \\
&\quad \times \frac{(-1)^{\alpha_1+\alpha_2+\alpha_3} q^{\frac{\alpha_1(\alpha_1+1)}{2} + \frac{\alpha_2(\alpha_2+1)}{2} + \frac{\alpha_3(\alpha_3+1)}{2} + \alpha_2(\alpha_1+\beta_1) + \alpha_3(\alpha_1+\beta_1+\alpha_2+\beta_2)}}{(q)_{\alpha_1}(q)_{\beta_1}(q)_{\alpha_2}(q)_{\beta_2}(q)_{\alpha_3}(q)_{\beta_3}} \\
&\quad \times \frac{(-1)^{\alpha_4} q^{(\alpha_2+\beta_2+1)\alpha_4} q^{\frac{\alpha_4(\alpha_4-1)}{2}}}{(q)_{\alpha_4}(q)_{\beta_4}} \\
&= -x^{-1} \sum_{\mathbf{d}} (-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} q^{\mathbf{l}^{\mathfrak{sl}_2} \cdot \mathbf{d}}}{(q)_{\mathbf{d}}},
\end{aligned}$$

where

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & -1 & 0 & -1 \\ 0 & 0 & -2 & -2 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{n} = (-1, -1, -2, -2, -1, -1, -1, -1),$$

$$\mathbf{l}^{\mathfrak{sl}_2} = \left(\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0\right).$$

Again, there are many different ways to apply Lemma 4.5 which produce slightly different but equivalent quivers.

Looking at the first few coefficients, we can easily upgrade this to the  $a$ -deformed version. Up to a prefactor, we have

$$F_{4_1}(x, a, q) = \sum_{\mathbf{d}} (-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} a^{\mathbf{a} \cdot \mathbf{d}} q^{\mathbf{l} \cdot \mathbf{d}}}{(q)_{\mathbf{d}}},$$

where

$$\mathbf{n} = (1, 1, 2, 2, 1, 1, 1, 1), \quad \mathbf{a} = (1, 0, 1, 0, 1, 0, 1, 0), \quad \mathbf{l} = \left(-\frac{3}{2}, 0, -\frac{3}{2}, 0, -\frac{3}{2}, 0, -\frac{3}{2}, 0\right).$$

5.2. **Quivers from  $R$ -matrices:  $\mathfrak{sl}_N$  case.** In order to extend [Par20b, Par21] and the discussion in Section 5.1 to symmetrically coloured  $\mathfrak{sl}_N$ , we need to find an expression for the  $R$ -matrix of the  $k$ -th symmetric representation of  $U_q(\mathfrak{sl}_N)$  similar to (66). In order to do this, we first briefly study  $U_q(\mathfrak{sl}_N)$  and its symmetric representations.

Recall that  $\mathfrak{sl}_N$  is the Lie algebra corresponding to the  $A_{N-1}$  root system. Using the usual dictionary order on  $\mathbb{C}^N$ , the simple roots of  $A_{N-1}$  are elements  $e_i - e_{i+1} \in \mathbb{C}^N$  where  $e_i$  is the vector with  $i$ -th entry 1 and all other entries 0. The Cartan matrix is

$$A_{ij} = \begin{cases} 2 & i = j, \\ -1 & |i - j| = 1, \\ 0 & |i - j| > 1, \end{cases}$$

and both  $\mathfrak{sl}_N$  and  $U_q(\mathfrak{sl}_N)$  are constructed from this via the Chevalley-Serre relations and their  $q$ -analogue. For each simple root  $\alpha_i = e_i - e_{i+1}$ , let  $\{X_i^+, X_i^-, K_i = q^{\frac{H_i}{2}}\}$  denote the quantum  $\mathfrak{sl}_2$  triple in  $U_q(\mathfrak{sl}_N)$ .

The  $k$ -th symmetric representation has a basis labelled<sup>12</sup> by  $\mathbf{a} = (a_1, \dots, a_{N-1})$  with  $k = a_0 \geq a_1 \geq \dots \geq a_{N-1} \geq a_N = 0$ , where the generators act as

$$X_i^+ |\mathbf{a}\rangle = [a_i - a_{i+1}]_q |\mathbf{a} - e_i\rangle, \quad X_i^- |\mathbf{a}\rangle = [a_{i-1} - a_i]_q |\mathbf{a} + e_i\rangle, \quad K_i |\mathbf{a}\rangle = q^{a_{i-1} + a_{i+1} - 2a_i} |\mathbf{a}\rangle.$$

We leave out the proof that this is actually a representation as this is an easy but tedious computation. A general form for  $U_q(\mathfrak{sl}_N)$   $R$  matrices is given in [Bur90] and we can find the corresponding specialisation to the  $k$ -th symmetric representation using the definitions of the representation above.

The  $R$ -matrix is an infinite summation over  $\mathbf{r} = r_i^j$  with  $1 \leq i \leq j \leq N - 1$  and each  $r_i^j$  is a non-negative integer. Define  $\mathbf{r}_{(i,j)}^{(k,l)}$  to be the vector  $(r_i^k, \dots, r_i^l, r_{i+1}^k, \dots, r_j^l)$  and denote

$$\mathbf{r}^j = \mathbf{r}_{(1,j)}^{(j,j)} = (r_1^j, \dots, r_j^j), \quad \mathbf{r}_j = \mathbf{r}_{(j,j)}^{(j,N-1)} = (r_j^j, \dots, r_j^{N-1}).$$

Finally, let  $|\cdot|$  denote the  $l^1$  norm which, as our vectors are non-negative, is simply the sum of the entries and replace all appearances of  $q^k$  by  $x$ . Then the  $R$ -matrix is (up to a constant prefactor):

$$R_{\mathfrak{sl}_N} |\mathbf{a}, \mathbf{b}\rangle = \sum_{\mathbf{r} > 0} \frac{(-1)^{|\mathbf{r}|} q^{C_N} x^{-\frac{1}{2}(a_1 + b_1 + |\mathbf{r}_1|)} (xq^{-b_1}; q^{-1})_{|\mathbf{r}_1|} (q^{a_1 - a_2 + |\mathbf{r}_2|}; q^{-1})_{|\mathbf{r}_1|}}{(q)_{\mathbf{r}}} \\ \times \prod_{j=2}^{N-1} (q^{b_{j-1} - b_j}; q^{-1})_{|\mathbf{r}_j|} (q^{a_j - a_{j+1} + |\mathbf{r}_{j+1}|}; q^{-1})_{|\mathbf{r}_j|} |\mathbf{b}', \mathbf{a}'\rangle,$$

where

$$C_N = \frac{1}{2} \mathbf{r} \cdot \mathbf{r} + \mathbf{a} \cdot M \cdot \mathbf{b} + \sum_{j=1}^{N-1} \frac{1}{4} |\mathbf{r}^j| (a_{j+1} + b_{j+1} - 2) - \frac{1}{4} \left( \sum_{i=2}^j r_i^j (a_{i-1} + b_{i-1}) \right) \\ + \sum_{i=1}^j r_i^j \left( |\mathbf{r}_{(i+1,j)}^{(j,N-1)}| + \frac{3}{4} (a_i - a_j) - \frac{1}{4} (b_i - b_j) \right), \\ M_{ij} = \begin{cases} 1 & i = j \\ -\frac{1}{2} & |i - j| = 1, \\ 0 & \text{else.} \end{cases} \quad a'_i = a_i - |\mathbf{r}_{(1,i)}^{(i,N-1)}|, \quad b'_i = b_i + |\mathbf{r}_{(1,i)}^{(i,N-1)}|.$$

While this formula is complicated, it is manageable for small values of  $N$  and in certain cases can be studied with  $N$  kept generic. It will produce a series in  $q$  and  $x^{-1}$ , as in order for

<sup>12</sup>The variables  $a_0$  and  $a_N$  are introduced for convenience.

the summand to be nonzero, we require  $a_1 > |\mathbf{r}_1|$ . The key point is that all  $q$  powers are quadratic, so for any positive braid knot this will give a quiver form in an identical manner to Theorem 1. The main difference is that all our variables and relations are  $N - 1$  dimensional vectors and we have a collection of extra free variables coming from the summations over  $\mathbf{r}$ 's. There is also an analogue of Theorem 2 and an extension of the inverse state sum method to  $\mathfrak{sl}_N$  but we do not discuss it here.

Finally, in order to match the above description up with (66), we define  $R_{\mathbf{a},\mathbf{b}}^{\mathbf{b}',\mathbf{a}'}$  by

$$(67) \quad R_{\mathbf{a},\mathbf{b}}^{\mathbf{b}',\mathbf{a}'} = \langle \mathbf{b}', \mathbf{a}' | R | \mathbf{a}, \mathbf{b} \rangle.$$

5.2.1. *The Trefoil.* The trefoil knot is given by the closure of the braid  $\sigma_1^3$ . We colour the top and bottom of the leftmost strand by  $\mathbf{a} = \mathbf{0}$  and close off the right hand strand. This means that we need to take the quantum trace over all possible labels  $\mathbf{b}$ . Analysing possibilities, we actually have no other free variables, as labels must always decrease going left to right over a positive crossing. Hence there are three  $R$ -matrices we need to consider. Going from bottom to top they are

$$R_{\mathbf{0},\mathbf{b}}^{\mathbf{b},\mathbf{0}}, \quad R_{\mathbf{b},\mathbf{0}}^{\mathbf{b},\mathbf{0}}, \quad R_{\mathbf{b},\mathbf{0}}^{\mathbf{0},\mathbf{b}}.$$

It turns out that for each of these cases, the summation over  $\mathbf{r}$  collapses to a single value. For  $R_{\mathbf{0},\mathbf{b}}^{\mathbf{b},\mathbf{0}}$  and  $R_{\mathbf{b},\mathbf{0}}^{\mathbf{0},\mathbf{b}}$ , the only nonzero terms corresponds to  $\mathbf{r} = 0$  and so

$$R_{\mathbf{0},\mathbf{b}}^{\mathbf{b},\mathbf{0}} = x^{-\frac{b_1}{2}}, \quad R_{\mathbf{b},\mathbf{0}}^{\mathbf{0},\mathbf{b}} = x^{-\frac{b_1}{2}}.$$

Similarly, for  $R_{\mathbf{b},\mathbf{0}}^{\mathbf{b},\mathbf{0}}$ , we have a collection of  $q$ -Pochhammers of the form  $(1; q^{-1})_{|\mathbf{r}_j|}$  for  $j > 1$ . These are 0 unless  $|\mathbf{r}_j| = 0$  so the only  $r_i^j$ 's which can be non zero are the  $r_1^j$ 's. Then the relation between  $\mathbf{b}$  and  $\mathbf{b}'$  forces  $r_1^j = b_j - b_{j+1}$ . After fixing this, we find that most of the  $q$ -Pochhammers are either 1 or cancel, and thus

$$R_{\mathbf{b},\mathbf{0}}^{\mathbf{b},\mathbf{0}} = (-1)^{b_1} q^{C_N} x^{-b_1} (x; q^{-1})_{b_1},$$

where

$$\begin{aligned} C_N &= \sum_{j=1}^{N-1} \frac{1}{2} (b_j - b_{j+1})^2 + \frac{1}{4} (b_j - b_{j+1})(b_{j+1} - 2) + \frac{3}{4} (b_j - b_{j+1})(b_1 - b_j) \\ &= \sum_{j=1}^{N-1} \frac{1}{4} (3b_1(b_j - b_{j+1}) - 2(b_j - b_{j+1}) - (b_j^2 - b_{j+1}^2)) = \frac{b_1^2 - b_1}{2}. \end{aligned}$$

Finally, we need to deal with the quantum trace over  $\mathbf{b}$ . In the general  $U_q(\mathfrak{sl}_N)$  case, we find that the trace factor will be  $q^{-|\mathbf{b}| - \frac{Nk}{2}}$ . We drop the term  $-\frac{Nk}{2}$  in the exponent (as it would correspond to a simple prefactor), which leaves us with the general formula for the trefoil:

$$F_{3_1}^{\mathfrak{sl}_N}(x, q) = \sum_{\mathbf{b}} (-1)^{b_1} q^{\frac{1}{2}(b_1-1)b_1} q^{-|\mathbf{b}|} x^{-2b_1} (x; q^{-1})_{b_1}.$$

We can switch from the series in  $x^{-1}$  to the one in  $x$  using Weyl symmetry  $x^{-1} \rightarrow ax$ :

$$F_{3_1}^{\mathfrak{sl}_N}(x, q) = \sum_{\mathbf{b}} (-1)^{b_1} q^{\frac{1}{2}(b_1-1)b_1} q^{-|\mathbf{b}|} a^{2b_1} x^{2b_1} (a^{-1}x^{-1}; q^{-1})_{b_1}.$$

In order to transform this back into the usual quiver form, we first perform the summations over the  $b_i$  variables for  $i > 1$ . Recalling that  $b_1 \geq b_2 \geq \dots \geq b_{N-1} \geq 0$ , we find that

$$\sum_{\mathbf{b}} q^{-|\mathbf{b}|} = \sum_{b_1=0}^{\infty} q^{b_1} a^{-b_1} \frac{(q^{b_1})_{N-2}}{(q)_{N-2}},$$

and so

$$F_{3_1}^{\text{sl}_N}(x, q) = \sum_{b_1=0}^{\infty} (-1)^{b_1} q^{\frac{1}{2}(b_1+1)b_1} a^{b_1} x^{2b_1} \frac{(q^{b_1})_{N-2} (a^{-1}x^{-1}; q^{-1})_{b_1}}{(q)_{N-2}}.$$

Next we can expand the  $q$ -Pochhammer  $(a^{-1}x^{-1}; q^{-1})_{b_1}$  as

$$(a^{-1}x^{-1}; q^{-1})_{b_1} = (a^{-1}x^{-1}q^{1-b_1}; q)_{b_1} = \sum_{i=0}^{b_1} (-1)^i q^{\frac{i(i-1)}{2}} \frac{(q)_b}{(q)_i (q)_{b-i}} x^{-i} a^{-i} q^{i-b_1 i}.$$

Substituting this in and letting  $b_1 = i + j$ , we get

$$F_{3_1}^{\text{sl}_N}(x, q) = \sum_{i,j}^{\infty} (-1)^j q^{\frac{1}{2}(2i+j+j^2)} a^j x^{i+2j} \frac{(q)_{i+j+N-2}}{(q)_{N-2} (q)_i (q)_j}.$$

To proceed, we make use of the general identity

$$\sum_{\alpha+\beta=d} (-1)^i \frac{q^{\frac{\alpha^2+\alpha}{2}}}{(q)_{\alpha} (q)_{\beta}} = 1.$$

This allows us to use Lemma 4.5 from [KRSS19] to get

$$\begin{aligned} F_{3_1}^{\text{sl}_N}(x, q) &= \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2} \frac{(-1)^{\alpha_2+\beta_2} q^{\frac{1}{2}(2(\alpha_1+\beta_1)+(\alpha_2+\beta_2)+(\alpha_2+\beta_2)^2)} a^{\alpha_2+\beta_2} x^{(\alpha_1+\beta_1)+2(\alpha_2+\beta_2)}}{(q)_{\alpha_1} (q)_{\alpha_2} (q)_{\beta_1} (q)_{\beta_2}} \\ &\quad \times (-q)^{\alpha_1+\alpha_2} q^{\frac{1}{2}(\alpha_1^2+\alpha_2^2)} q^{(N-2)\alpha_1+\alpha_2(N-2+\alpha_1+\beta_1)} q^{-\frac{1}{2}(\alpha_1+\alpha_2)} \\ &= \sum_{\mathbf{d}=(d_1, d_2, d_3, d_4)} \frac{(-1)^{d_1+d_4} q^{\frac{1}{2}\mathbf{d} \cdot C \cdot \mathbf{d} + \frac{1}{2}(-d_1-2d_1+2d_3+d_4)} a^{d_1+2d_2+d_4} x^{(d_1+d_3)+2(d_2+d_4)}}{(q)_{d_1} (q)_{d_2} (q)_{d_3} (q)_{d_4}} \end{aligned}$$

with

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Up to a conventional choice (which can be fixed by  $x \rightarrow qx$ ) and reordering variables, this is exactly the  $a$ -deformed quiver from Section 5.1.1.

In principle, this can be applied to other positive braid knots, though the analysis will be more complicated for generic  $N$ . Fixing a small value of  $N$  though this will easily produce  $F_K^{\text{sl}_3}$ ,  $F_K^{\text{sl}_4}$ ,  $F_K^{\text{sl}_5}$ ,  $\dots$  invariants and quiver forms for this class of knots.

**5.3.  $R$ -matrices and braiding from brane configurations.** The matrix elements (66) and (67) have a natural physical interpretation in the brane setup of Section 2.2. If a knot (link)  $K$  is represented as the closure of a braid  $\beta$ , each elementary transformation (crossing) represented by a generator of the braid group corresponds to an elementary interface (domain wall)  $\mathcal{D}_i$  in the direction along which the braid is stretched. Following [CGR15, GNS+16], we parametrize this direction by  $\sigma \in S^1$ . When both the rank and the dimensions of representations colouring the strands are finite-dimensional, the interfaces can be described by matrix factorizations in topological Landau-Ginzburg models (see Figure 23 in [CGR15] or Figure 9 in [GNS+16]).

Our setup in this paper is similar, except that strands carry infinite-dimensional representations, namely the Verma modules of  $U_q(\mathfrak{sl}_N)$ . In the brane configuration, this corresponds to replacing M2-branes – that produce strands coloured by finite-dimensional representations – with M5-branes. The two are related: specializing  $x = q^n$  corresponds to the partial Higgsing implemented in the brane geometry by a version of the Hanany-Witten effect that creates  $n$  M2-branes stretched between M5 and M5' branes (see loc. cit. and references therein). Since in this paper we are mostly interested in infinite-dimensional representations with arbitrary complex weights  $\log x_i$ , the relevant brane configuration (3) involves two sets of fivebranes:  $N$  M5-branes that produce  $SL(N, \mathbb{C})$  Chern-Simons theory on  $S^3$ , and additional M5'-branes supported on  $L_K$  that produce the strands of  $K = L_K \cap S^3$  coloured by infinite-dimensional representations with highest weights  $x_i \in \mathbb{C}^*$ ,  $i = 1, \dots, \rho$ .<sup>13</sup>

In order to see how the  $R$ -matrices and braiding are realized from the perspective of 3d-3d correspondence, we need to focus on the  $\mathbb{R}^2 \times S^1$  part of the spacetime in (3). This is the three-dimensional space where 3d  $\mathcal{N} = 2$  theory  $T[M_3]$  “lives.” Many aspects of this 3d  $\mathcal{N} = 2$  theory are understood quite well. It contains  $T[S^3]$ , which is a relatively simple theory. The additional degrees of freedom come from a codimension-two BPS defect in the 6d (0, 2) theory on M5-branes. According to (3), the support of the 6d (0, 2) theory is  $\mathbb{R}^2 \times S^1 \times S^3$ , whereas the support of a codimension-two BPS defect is  $\mathbb{R}^2 \times S^1 \times K$ .<sup>14</sup> As before, we consider  $K$  represented by a closure of a braid stretched, say, along the great circle of  $S^3$  (see Figure 23 in [CGR15] or Figure 9 in [GNS+16]). Then the two sets of the fivebranes in (3) share a total of four dimensions: the three dimensions of  $\mathbb{R}^2 \times S^1$  and one dimension of  $S^1 \subset S^3$ , parametrized by  $\sigma$ .

The degrees of freedom of this 4-dimensional theory are precisely the degrees of freedom of the codimension-two defect in 6d (0, 2) theory. The 3d  $\mathcal{N} = 2$  theory  $T[M_K]$  is the result of compactifying this 4d theory on the great circle  $S^1 \subset S^3$ . This compactification is not entirely trivial, though, because the strands of  $K$  are braided as  $\sigma \in S^1$  is traversed. One can imagine that the 4d theory is piecewise constant and undergoes sharp transitions where nontrivial braiding occurs (see Figure 8). These are precisely the codimension-one interfaces representing the  $R$ -matrices, and the theory  $T[M_K]$  is basically a composition of these interfaces.

The individual interfaces representing braiding and  $R$ -matrices preserve 4 real supersymmetries, just like the 3d  $\mathcal{N} = 2$  theory  $T[M_K]$  itself. Along the  $\mathbb{R}^2$  directions of  $\mathbb{R}^2 \times S^1$

<sup>13</sup>In a sense, the most generic case is when  $\rho = N$ . In this case, the two sets of fivebranes in (3) can be reconnected in single smooth configuration supported on the Lagrangian submanifold

$$(S^3 \setminus \nu K) \cup_{T^2} (L_K \setminus \nu K)$$

where, as above,  $K = L_K \cap S^3$ .

<sup>14</sup>Note that  $L_K$  is non-compact. If it was compact, 3d  $\mathcal{N} = 2$  theory would also contain a non-trivial sector associated with the modes of 6d (0, 2) theory compactified on  $L_K$ .

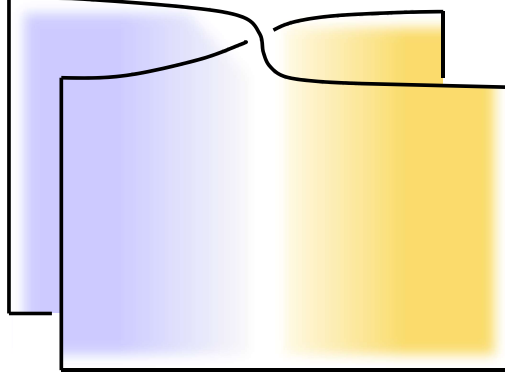


FIGURE 8. A half-BPS interface interpolating between two vacua of the theory on a stack of branes that represents the  $R$ -matrix / braiding. Evaluating the partition function of the combined system in the Omega-background along  $\mathbb{R}_q^2$  gives the matrix elements (66) and (67).

there is Omega-background with the equivariant parameter  $q$ . Therefore, the matrix elements (66) and (67) have a physical interpretation in terms of  $\mathbb{R}_q^2 \times S^1 \times \mathbb{R}$  partition function of the 4-dimensional defect in 6d (0,2) theory with two different vacua at  $+\infty$  and  $-\infty$  along the  $\mathbb{R}$ -direction. One can reduce this problem to the study of interfaces in a 3d QFT by compactifying on the  $S^1$  or a circle in  $\mathbb{R}_q^2$ , such that  $\mathbb{R}_q^2/U(1) \cong \mathbb{R}_+$ .

**5.4. Quivers from inverted Habiro series.** There is a slightly different way to get quivers for the trefoil and the figure-eight knot. That is to use the *inverted Habiro expansion* studied in [Par21]. As briefly explained in Section 3.3.3, inverted Habiro expansion of  $F_K$  is the series

$$F_K(x, q) = - \sum_{m=1}^{\infty} \frac{a_{-m}(K)}{\prod_{j=0}^{m-1} (x + x^{-1} - q^j - q^{-j})},$$

where  $a_{-m}(K)$  denotes a natural extension of Habiro's cyclotomic coefficients to negative direction. When the knot is either the trefoil or the figure-eight knot,  $a_m(\mathbf{4}_1) = 1$  and  $a_m(\mathbf{3}_1^r) = (-1)^m q^{\frac{m(m+3)}{2}}$ , so they can be extended to negative  $m$  in a straightforward manner. We will use these expressions to find the corresponding quiver forms.

**5.4.1. Right-handed trefoil.** As a power series in  $x$ ,

$$\begin{aligned} F_{\mathbf{3}_1^r}(x, q) &= q \sum_{n \geq 0} \frac{(-1)^n q^{-\frac{n(n-1)}{2}}}{\prod_{j=0}^n (x + x^{-1} - q^j - q^{-j})} = \frac{qx}{1-x} \sum_{i, k \geq 0} (-1)^i q^{-\frac{i(i-1)}{2}} q^{-ki} \begin{bmatrix} 2i+k \\ k \end{bmatrix}_q x^{i+k} \\ &= \frac{qx}{1-x} \sum_{i, k_1, k_2 \geq 0} \frac{(-1)^{i+k_1} q^{-\frac{i(i-1)}{2} + \frac{k_1(k_1+1)}{2} + (k_1-k_2)i}}{(q)_{k_1} (q)_{k_2}} x^{i+k_1+k_2} \\ &= qx \sum_{i_1, i_2, j_1, j_2, k_1, k_2 \geq 0} \frac{(-1)^{i_2+j_1+k_1} q^{-\frac{i_2(i_2-1)}{2} - i_1 i_2 + i_1 + \frac{j_1(j_1+1)}{2} + \frac{k_1(k_1+1)}{2} + (i_1+i_2)(k_1-k_2)}}{(q)_{i_1} (q)_{i_2} (q)_{j_1} (q)_{j_2} (q)_{k_1} (q)_{k_2}} \\ &\quad \times x^{i_1+i_2+j_1+j_2+k_1+k_2}. \end{aligned}$$

This can be written as

$$F_{\mathbf{3}_1}(x, q) = qx \sum_d (-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} q^{\mathbf{l}^{\text{sl}_2} \cdot \mathbf{d}}}{(q)_d},$$

where

$$C = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{n} = (1, 1, 1, 1, 1, 1),$$

$$\mathbf{l}^{\text{sl}_2} = \left(-1, \frac{1}{2}, -\frac{3}{2}, 0, -\frac{3}{2}, 0\right).$$

Upgrading this to the  $a$ -deformed version,

$$\mathbf{a} = (1, 0, 1, 0, 1, 0), \quad \mathbf{l} = \left(-1, \frac{1}{2}, -\frac{3}{2}, 0, -\frac{3}{2}, 0\right).$$

5.4.2. *Figure-eight knot.* As a power series in  $x$ ,

$$\begin{aligned} F_{\mathbf{4}_1}(x, q) &= - \sum_{n \geq 0} \frac{1}{\prod_{j=0}^n (x + x^{-1} - q^j - q^{-j})} = - \frac{x}{1-x} \sum_{i, k \geq 0} q^{-ki} \begin{bmatrix} 2i+k \\ k \end{bmatrix}_q x^{i+k} \\ &= - \frac{x}{1-x} \sum_{i, k_1, k_2 \geq 0} \frac{(-1)^{k_1} q^{\frac{k_1(k_1+1)}{2} + (k_1 - k_2)i}}{(q)_{k_1} (q)_{k_2}} x^{i+k_1+k_2} \\ &= -x \sum_{i_1, i_2, j_1, j_2, k_1, k_2 \geq 0} \frac{(-1)^{i_1+j_1+k_1} q^{\frac{i_1(i_1+1)}{2} + \frac{j_1(j_1+1)}{2} + \frac{k_1(k_1+1)}{2} + (i_1+i_2)(k_1-k_2)}}{(q)_{i_1} (q)_{i_2} (q)_{j_1} (q)_{j_2} (q)_{k_1} (q)_{k_2}} \\ &\quad \times x^{i_1+i_2+j_1+j_2+k_1+k_2} \\ &= -x \sum_d (-q^{\frac{1}{2}})^{\mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{x^{\mathbf{n} \cdot \mathbf{d}} q^{\mathbf{l}^{\text{sl}_2} \cdot \mathbf{d}}}{(q)_d}, \end{aligned}$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{n} = (1, 1, 1, 1, 1, 1),$$

$$\mathbf{l}^{\text{sl}_2} = \left(\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0\right).$$

Rearranging the nodes, we can write

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{n} = (1, 1, 1, 1, 1, 1),$$

$$\mathbf{l}^{\text{sl}_2} = \left(0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

which is the same as the quiver (39) that was found empirically. We have just derived this quiver! Upgrading this to the  $a$ -deformed version,

$$\mathbf{a} = (0, 0, 0, 1, 1, 1), \quad \mathbf{l} = \left(0, 0, 0, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}\right).$$

## 6. QUANTIZED COULOMB BRANCH OF THE 3D-5D SYSTEM

In the physical realization of the HOMFLY-PT polynomials, the variable  $a$  is identified with the Kähler parameter of the resolved conifold geometry  $X$  or, equivalently, with the Coulomb branch parameter of the 5d  $\mathcal{N} = 2$  gauge theory “engineered” [KKV97] by compactification on the resolved conifold. In the standard string terminology, it is a *closed string* modulus because it describes the geometry of the background with no branes.

Similarly, in the M-theory setup of [OV00], the variable  $x$  is identified with the open string modulus or, more precisely, with the modulus of the fivebranes supported on a Lagrangian submanifold in  $X$ . In five spacetime dimensions complementary to  $X$ , these fivebranes span a three-dimensional subspace. In other words, they represent a codimension-two defect (a.k.a. a surface operator) in the 5d  $\mathcal{N} = 2$  gauge theory. Following [GW08], this half-BPS surface operator can be conveniently described by coupling a 3d  $\mathcal{N} = 2$  theory – in fact, precisely the  $T[S^3 \setminus K]$  theory – to the 5d  $\mathcal{N} = 2$  theory [DGH11, Guk16]. Then the variable  $x$  can be understood as the Coulomb branch parameter of the 3d  $\mathcal{N} = 2$  theory, much like  $a$  is the Coulomb branch parameter of the 5d  $\mathcal{N} = 2$  theory.

Both variables  $x$  and  $a$  have their corresponding “conjugates”,  $y$  and  $a^D$ , respectively, such that turning on the Omega-background in the coupled 3d-5d system has the effect of quantizing the holomorphic symplectic phase space parametrized by  $(a, a^D)$  in the case of 5d theory and parametrized by  $(x, y)$  in the case of 3d theory.<sup>15</sup> In particular, in the presence of surface operators, the Nekrasov partition function [Nek03, NY03] computes the K-theoretic instanton-vortex partition function of the coupled system, with the asymptotic behaviour

$$Z_{3d/5d} \simeq \exp \left( -\frac{1}{\hbar^2} \mathcal{F}(a) + \frac{1}{\hbar} \widetilde{W}(a, x) + \dots \right).$$

If we identify  $\hbar = g_s$ , it has the familiar form of the “closed + open” topological string partition function, with the small but important caveat that all variables are  $\mathbb{C}^*$ -valued, associated with the K-theoretic lift of the instanton / vortex counting. In particular, the holomorphic symplectic forms relevant to 3d and 5d Coulomb branches are, respectively,

$$d \log x \wedge d \log y, \quad d \log a \wedge d \log a^D.$$

Therefore, quantization with respect to these holomorphic symplectic forms on the space  $(\mathbb{C}^*)^4$  parametrized by  $(x, y, a, a^D)$  replaces the algebra of functions by the algebra of operators that obey the following  $q$ -commutation relations:

$$(68) \quad \hat{y}\hat{x} = q\hat{x}\hat{y}, \quad \hat{b}\hat{a} = q\hat{a}\hat{b},$$

where we introduced a new notation  $b \equiv a^D$  to make the parallel with  $A$ -polynomial more manifest. In 3d/5d coupled system, these operators can be understood as line operators. Their various combinations which represent Ward identities are certain  $q$ -difference operators

<sup>15</sup>Note, both ingredients can exist independently. For example, one can consider a trivial 5d theory, in which case the coupled 3d-5d system is nothing but the 3d standalone theory. And, similarly, one can consider a trivial 3d theory, in which case there is no surface operator in 5d theory.

that annihilate the K-theoretic vortex-instanton partition function, see e.g. [MM12, ABM<sup>+</sup>15, AKM<sup>+</sup>18, JLN21].

**6.1. The holomorphic Lagrangian subvariety.** From the physical discussion at the start of this section, it is clear that classically the Coulomb branch of the 3d-5d coupled system is a holomorphic Lagrangian  $\Gamma_K$  in  $(\mathbb{C}^*)^4$  endowed with the holomorphic symplectic form<sup>16</sup>

$$(69) \quad \boxed{\Omega := d \log x \wedge d \log y + d \log a \wedge d \log b.}$$

Naturally, the projection of  $\Gamma_K$  on  $(\mathbb{C}^*)_{x,y,a}^3$  is the zero locus of the  $a$ -deformed  $A$ -polynomial. In this sense,  $\Gamma_K$  can be thought of as the holomorphic Lagrangian lift of the  $A$ -polynomial. Note that such holomorphic Lagrangian is uniquely determined, as we can solve for  $b$  as a function of  $a$  and  $x$  by

$$b = \exp \left( \int \frac{\partial \log y}{\partial \log a} d \log x \right).$$

Away from the discriminant locus, different branches of  $y$  will lead to different branches of  $b$ , and they can be understood as the expectation values of the corresponding operators ( $\hat{y}$  and  $\hat{b}$ ) acting on  $F_K$  for those branches that we discussed in Section 3.

A novel feature of this holomorphic Lagrangian is that instead of projecting it to  $(\mathbb{C}^*)_{x,y,a}^3$ , we can project it down to other hyperplanes such as  $(\mathbb{C}^*)_{a,b,x}^3$ , and study the polynomial defining the locus. This particular polynomial (the one we get upon projection to  $(\mathbb{C}^*)_{a,b,x}^3$ ) will be called the  $B$ -polynomial of the knot  $K$  and is the subject of the next subsection. To recap, the zero sets of the  $A$ - and  $B$ -polynomials are simply the projections of  $\Gamma_K$ , so we have the following diagram.

$$\begin{array}{ccc}
 & \Gamma_K = Z(AB_K) & \\
 \swarrow \pi_b & & \searrow \pi_y \\
 Z(A_K) & & Z(B_K) \\
 \searrow \pi_y & & \swarrow \pi_b \\
 & (\mathbb{C}^*)_{x,a}^2 & 
 \end{array}$$

We call the ideal defining  $\Gamma_K$  the  $AB$ -ideal and denote it by  $AB_K$  for an obvious reason; it unifies  $A$ - and  $B$ -polynomials.

Another notable feature of  $\Gamma_K$  is that it enjoys Weyl symmetry. This is because  $\Gamma_K$  describes the semiclassical behaviour of  $F_K$  which enjoys Weyl symmetry. Recall that on the  $A$ -polynomial (or on  $F_K$ ), the Weyl symmetry acts on the variables  $x$ ,  $y$  and  $a$  as

$$x \mapsto a^{-1}x^{-1}, \quad y \mapsto y^{-1}, \quad a \mapsto a.$$

The action of the Weyl symmetry on the variable  $b$  (dual to  $a$ ) can be deduced as follows.

**Proposition 3.** *Under the Weyl symmetry,  $b$  transforms in the following way:*

$$b \mapsto y^{-1}b.$$

<sup>16</sup>One can see this also from (74).

*Proof.* This can be most easily seen by making the following change of variables:

$$u_1 = ax, \quad u_2 = x.$$

Let  $v_1, v_2$  be the variables dual to  $u_1, u_2$ , respectively. That is,  $v_1 = b$  and  $v_2 = b^{-1}y$ . Then the Weyl symmetry acts by

$$u_1 \mapsto u_2^{-1}, \quad u_2 \mapsto u_1^{-1}$$

on  $u_1$  and  $u_2$ . Since  $v_1$  and  $v_2$  are variables dual to  $u_1$  and  $u_2$ , the Weyl symmetry should act on them by

$$v_1 \mapsto v_2^{-1}, \quad v_2 \mapsto v_1^{-1}.$$

Therefore, if we denote the Weyl symmetry map by  $W$ , we have

$$W(b) = W(v_1) = v_2^{-1} = y^{-1}b.$$

□

Consequently, the holomorphic Lagrangian is preserved under the action of the Weyl symmetry  $W$ :

$$W(\Gamma_K) = \Gamma_K.$$

Just like the  $A$ -polynomial can be quantized into a  $q$ -difference operator annihilating  $F_K$ , the holomorphic Lagrangian itself can be quantized. The quantum  $AB$ -ideal (quantization of the  $AB$ -ideal) is a left ideal of  $q$ -difference operators in  $\hat{x}, \hat{y}, \hat{a}, \hat{b}$ , which act on  $F_K(x, a, q)$  by

$$(70) \quad \begin{aligned} \hat{x}F_K(x, a, q) &= xF_K(x, a, q), & \hat{a}F_K(x, a, q) &= a F_K(x, a, q), \\ \hat{y}F_K(x, a, q) &= F_K(qx, a, q), & \hat{b}F_K(x, a, q) &= F_K(x, qa, q). \end{aligned}$$

This quantum ideal annihilates  $F_K$  regardless of the choice of branch. We will see some explicit examples in Section 6.3.

**Remark 10.** While we primarily work with  $F_K$ , we could have chosen knot conormal instead of knot complement as the Lagrangian filling. Then the wave function we get would be the coloured HOMFLY-PT generating function, written as a function of  $y, a, q$ . The holomorphic Lagrangian  $\Gamma_K$ , however, does not change, since it comes from a theory at infinity, which specializes to the curve count of knot contact homology that gives the  $A$ -polynomial, and that is independent of the choice of filling.

6.1.1. *Asymptotic behavior of the holomorphic Lagrangian near the boundary.* Let  $(\mathbb{C}^*)^4$  be the space parametrized by  $x, y, a, b \in \mathbb{C}^*$  and equipped with the holomorphic symplectic form

$$\Omega = d \log x \wedge d \log y + d \log a \wedge d \log b.$$

Let  $\Gamma_K \subset (\mathbb{C}^*)^4$  be the holomorphic Lagrangian associated to a knot  $K$ .

In this subsection, we are interested in studying the asymptotic behavior of  $\Gamma_K$  near the boundary of  $\mathbb{C}^*$  where we can talk about various branches of  $\Gamma_K$  (and therefore the corresponding wave functions  $F_K$ ) without ambiguity.

Asymptotically near the boundary,  $\Gamma_K$  should look like a 2-dimensional linear subspace  $V$  of  $\mathbb{C}^4$  (parametrized by logarithmic variables) that is holomorphic Lagrangian. Suppose that this linear subspace is described by

$$x^{k_i} y^{l_i} a^{m_i} b^{n_i} = 1, \quad i = 1, 2.$$

We are assuming that the two vectors  $\mathbf{k}_i := (k_i, l_i, m_i, n_i)$  are independent. By eliminating some coordinates, we may choose to work with two vectors such that  $n_1 = 0$  and  $l_2 = 0$ .

Then the first equation determines the slope of the branch of  $A_K$ , and the second equation determines that of  $B_K$ .

The Lagrangian condition of this linear subspace can be described in a simple way.

**Proposition 4.** *The above linear subspace  $V$  is Lagrangian iff*

$$\det\begin{pmatrix} k_1 & l_1 \\ k_2 & l_2 \end{pmatrix} + \det\begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} = 0.$$

*Proof.* The symplectic form  $\Omega$  can be expressed as the following matrix:

$$S := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

By definition,  $V$  is Lagrangian iff  $v_1^T S v_2 = 0$  for any choice of basis  $\{v_1, v_2\} \subset V$ . Moreover, a vector  $v$  is in  $V$  iff  $\mathbf{k}_i^T v = 0$  for  $i = 1, 2$ . From this, it is easy to see that  $V$  is Lagrangian iff  $\mathbf{k}_1^T S \mathbf{k}_2 = 0$ , and this is exactly the equation written above.  $\square$

A direct application of this simple fact is the following. Suppose that  $n_1 = 0$  and  $l_2 = 0$  so that

$$x^{k_1} y^{l_1} a^{m_1} = 1, \quad x^{k_2} a^{m_2} b^{n_2} = 1.$$

Then the Lagrangian condition is simply  $\frac{l_1}{m_1} = \frac{n_2}{k_2}$ . That is, for any branch, the slope of the Newton polygon of  $A_K$  as a polynomial in  $y$  and  $a$  equals that of  $B_K$  as a polynomial in  $b$  and  $x$ . Below, we check this in a few explicit examples.

**Example 1** (Trefoil, right-handed). We write  $A_K$  and  $B_K$  in matrix form. For  $A_K$  (resp.  $B_K$ ), horizontal direction corresponds to  $a$ -degree (resp.  $x$ -degree) and the vertical direction corresponds to  $y$ -degree (resp.  $b$ -degree). The degrees get bigger going up and right.

$$A_{\mathbf{3}_1} = \begin{pmatrix} 0 & 0 & -x^3 & x^4 \\ -1 & 2x^2+x & -x^4+x^3-2x^2 & 0 \\ 0 & 1-x & 0 & 0 \end{pmatrix}$$

$$B_{\mathbf{3}_1} = \begin{pmatrix} 0 & 0 & a^2-a^3 & a^4-a^3 \\ -1 & a & a-2a^2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

**Example 2** (Figure-eight).

$$A_{\mathbf{4}_1} = \begin{pmatrix} 0 & -x^2 & 2x^3 & -x^4 & 0 \\ 1 & -3x & 2x^4+2x^2 & -3x^5 & x^6 \\ 0 & -2x^2+3x-1 & 0 & -x^6+3x^5-2x^4 & 0 \\ 0 & 0 & x^4-2x^3+x^2 & 0 & 0 \end{pmatrix}$$

$$B_{\mathbf{4}_1} = \begin{pmatrix} 0 & -a^3+2a^2-a & 2a^4-4a^3+2a^2 & -a^5+2a^4-a^3 & 0 \\ 1-a & 3a^2-3a & -2a^3+4a^2-2a & 3a^2-3a^3 & a^4-a^3 \\ 0 & 2-a & -3a & 2a^2-a & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

**Example 3** ( $\mathbf{5}_2$ ).

$A_{\mathbf{5}_2} =$

$$\begin{pmatrix} -1 & 3x & -3x^2 & x^3 & 0 & 0 & 0 & 0 \\ 0 & 3x^2-4x+2 & x^4-4x^3+5x^2-4x & -4x^5-x^4+2x^3+2x^2 & 6x^6+2x^5-3x^4 & -4x^7 & x^8 & 0 \\ 0 & 0 & -3x^4+8x^3-8x^2+4x-1 & -3x^6+2x^5-x^4+x^3+x & 8x^7-x^6-4x^5+x^4-4x^3 & -8x^8+x^7+x^6+6x^5 & 4x^9-4x^7 & x^9-x^{10} \\ 0 & 0 & 0 & x^6-4x^5+6x^4-4x^3+x^2 & 3x^8-4x^7-x^6+2x^5 & -4x^9+5x^8+2x^7-3x^6 & 2x^{10}-4x^9+2x^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x^{10}+3x^9-3x^8+x^7 & 0 & 0 \end{pmatrix}$$

$B_{\mathbf{5}_2} =$

$$\begin{pmatrix} -a^3+3a^2-3a+1 & 3a^4-9a^3+9a^2-3a & -3a^5+9a^4-9a^3+3a^2 & a^6-3a^5+3a^4-a^3 & 0 & 0 & 0 & 0 \\ 0 & -3a^2+6a-3 & 7a^3-15a^2+9a-1 & -5a^4+14a^3-13a^2+4a & a^5-8a^4+13a^3-6a^2 & 4a^5-8a^4+4a^3 & -a^6+2a^5-a^4 & 0 \\ 0 & 0 & a^2-4a+3 & -a^3+5a^2-7a+3 & -3a^3+11a^2-8a & 2a^4-10a^3+8a^2 & 4a^4-4a^3 & a^4-a^5 \\ 0 & 0 & 0 & a-1 & a-3 & 4a-a^2 & -2a^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In all the examples above, not just the slopes but the whole Newton polygons match up. We conjecture that this is true in general:

**Conjecture 7.** *The Newton polygon of  $A_K$  as a polynomial in  $y$  and  $a$  agrees with that of  $B_K$  as a polynomial in  $b$  and  $x$ .*

## 6.2. $B$ -polynomial.

6.2.1. *Quantum  $B$ -polynomial.* In [MM21], it was observed that the cyclotomic coefficients of the coloured HOMFLY-PT polynomials satisfy not only  $q$ -difference equations in the colour variable but also  $q$ -difference equations in the variable  $a = q^N$ . The  $q$ -holonomicity of the cyclotomic coefficients implies that the  $\mathfrak{sl}_N$  coloured Jones polynomials are  $q$ -holonomic in the variable  $a = q^N$ .

Let us recall that the operators  $\hat{a}, \hat{b}$  and  $\hat{x}, \hat{y}$  act on  $\mathfrak{sl}_N$  coloured Jones polynomials by

$$\begin{aligned} \hat{x} J_r^{\mathfrak{sl}_N}(K; q) &= q^r J_r^{\mathfrak{sl}_N}(K; q), & \hat{a} J_r^{\mathfrak{sl}_N}(K; q) &= q^N J_r^{\mathfrak{sl}_N}(K; q), \\ \hat{y} J_r^{\mathfrak{sl}_N}(K; q) &= J_{r+1}^{\mathfrak{sl}_N}(K; q), & \hat{b} J_r^{\mathfrak{sl}_N}(K; q) &= J_r^{\mathfrak{sl}(N+1)}(K; q). \end{aligned}$$

We can see that  $(\hat{a}, \hat{b})$  interact with the group rank  $N$  in complete analogy with the action of  $(\hat{x}, \hat{y})$  on the representation  $r$ ; the commutation relations are as in (68).

As pointed out above, the  $\mathfrak{sl}_N$  coloured Jones polynomials satisfy a recurrence relation in variable  $N$ . We will call the corresponding  $q$ -difference operator the *quantum  $B$ -polynomial* and denote it by  $\hat{B}_K(\hat{a}, \hat{b}, x, q)$ . In other words, the recurrence relation in  $N$  is given by

$$\hat{B}_K(\hat{a}, \hat{b}, x, q) J_r^{\mathfrak{sl}_N}(K; q) = 0.$$

Note, this is in analogy with the quantum  $A$ -polynomials which annihilate  $\mathfrak{sl}_N$  coloured Jones polynomials acting with  $\hat{x}$  and  $\hat{y}$ . Likewise, the quantum  $B$ -polynomial annihilates  $F_K(x, a, q)$  associated to any branch:

$$\hat{B}_K(\hat{a}, \hat{b}, x, q) F_K^{(\alpha)}(x, a, q) = 0.$$

The action of  $\hat{a}$  and  $\hat{b}$  on  $F_K$  were described in (70).

6.2.2. *Brief comment on normalisations.* Before presenting some  $B$ -polynomials, we should briefly pause to comment on how the different normalisations of  $F_K$  and  $J_r^{\mathfrak{sl}_N}$  discussed in Section 2.3.3 affect the  $B$ -polynomial. Observing that

$$\begin{aligned} \hat{b} \frac{(xq; q)_\infty}{(xa; q)_\infty} &= (1 - xa) \frac{(xq; q)_\infty}{(xa; q)_\infty} \hat{b}, \\ \hat{b} e^{-\frac{\log(x) \log(a)}{2\hbar}} x^{\frac{1}{2}} \frac{(a; q)_\infty (xq; q)_\infty}{(xa; q)_\infty (q; q)_\infty} &= \frac{1 - xa}{x^{\frac{1}{2}} (1 - a)} e^{-\frac{\log(x) \log(a)}{2\hbar}} x^{\frac{1}{2}} \frac{(a; q)_\infty (xq; q)_\infty}{(xa; q)_\infty (q; q)_\infty} \hat{b}, \end{aligned}$$

we see that given a recursion relation  $\hat{B}(\hat{b}, \hat{a}, x, q)$  for a reduced invariant, the corresponding relations for the unreduced and fully unreduced invariants are

$$\begin{aligned} \hat{B}^{\text{unreduced}}(\hat{a}, \hat{b}, x, q) &= \hat{B} \left( \hat{a}, \frac{1}{1 - x\hat{a}} \hat{b}, x, q \right), \\ \hat{B}^{\text{fully unreduced}}(\hat{a}, \hat{b}, x, q) &= \hat{B} \left( \hat{a}, \frac{x^{\frac{1}{2}}(1 - \hat{a})}{1 - x\hat{a}} \hat{b}, x, q \right). \end{aligned}$$

When making these substitutions for the  $\hat{b}$  operator, we also usually rescale by left multiplication so that the coefficients of  $\hat{b}^n$  are polynomials in  $\hat{a}, x, q$  with no common factors.

Looking at the unknot, the reduced  $B$ -polynomial is easily seen to be equal to

$$B_{\mathbf{0}_1}(\hat{a}, \hat{b}, x, q) = \hat{b} - 1,$$

so the corresponding unreduced and fully unreduced  $B$ -polynomials are given by

$$(71) \quad \begin{aligned} B_{\mathbf{0}_1}^{\text{unreduced}}(\hat{a}, \hat{b}, x, q) &= \hat{b} - (1 - x\hat{a}) \\ B_{\mathbf{0}_1}^{\text{fully unreduced}}(\hat{a}, \hat{b}, x, q) &= x^{\frac{1}{2}}(1 - \hat{a})\hat{b} - (1 - x\hat{a}). \end{aligned}$$

Similarly, if we compute the  $B$ -polynomial using quiver forms and want to include a prefactor as in Equation (17), then  $\hat{b}$  acts on this prefactor as

$$\hat{b} \exp\left(\frac{p(\log x, \log a)}{\hbar}\right) = \exp\left(\frac{p(\log x, \hbar + \log a)}{\hbar}\right) \hat{b}.$$

Defining

$$p'(\log(x), \log(a)) := p(\log x, \log a) - p(\log x, \hbar + \log a),$$

we see that including the prefactor modifies the  $B$  polynomial by

$$(72) \quad \hat{B}(\hat{b}, \hat{a}, x, q) \mapsto \hat{B}\left(\exp\left(\frac{p'(\log x, \log a)}{\hbar}\right) \hat{b}, \hat{a}, x, q\right).$$

Overall, we see that with a little care it is easy to adjust the  $B$ -polynomial to various conventions and normalisations.

Quantum  $B$ -polynomials for simple knots in the reduced normalisation are given in Table 1.

$K$	$\hat{B}_K(\hat{a}, \hat{b}, x, q)$
$0_1$	$1 - \hat{b}$
$3_1$	$qx^2 - x(1 + q - (1 + qx)\hat{a} + qx^2\hat{a}^2)\hat{b} + (1 - \hat{a})(1 - qx\hat{a})\hat{b}^2$
$4_1$	$q^2x^2\hat{a}^2 + qx\hat{a}(1 + q - (1 + 3qx + q^2x^2)\hat{a} + qx^2(1 + q)\hat{a}^2)\hat{b}$ $+ (1 - \hat{a})(1 - qx\hat{a})(1 - 2qx(1 + qx)\hat{a} + q^3x^3\hat{a}^2)\hat{b}^2$ $- x(1 - \hat{a})(1 - q\hat{a})(1 - qx\hat{a})(1 - q^2x\hat{a})\hat{b}^3$
$5_1$	$-qx^4(1 + q + q^2 - (1 + q)(1 + qx)\hat{a} + qx(1 + x + qx)\hat{a}^2 - qx^2(1 + qx)\hat{a}^3 + q^2x^4\hat{a}^4)\hat{b}$ $+ x^2(1 - \hat{a})(1 - qx\hat{a})(1 + q + q^2 - q(1 + qx)\hat{a} + q^2x^2(1 + q)\hat{a}^2)\hat{b}^2$ $- (1 - \hat{a})(1 - q\hat{a})(1 - qx\hat{a})(1 - q^2x\hat{a})\hat{b}^3 + q^3x^6$

TABLE 1. Quantum  $B$ -polynomials for some simple knots.

6.2.3. *Classical  $B$ -polynomial.* Similarly to the case of  $A$ -polynomial, the  $q \rightarrow 1$  limit of the quantum  $B$ -polynomial can be obtained directly from the effective twisted superpotential:

$$(73) \quad \lim_{q \rightarrow 1} \hat{B}_K(\hat{a}, \hat{b}, x, q) = B_K(a, b, x) = 0 \quad \Leftrightarrow \quad \log b = \frac{\partial \widetilde{\mathcal{W}}_{T[M_K]}^{\text{eff}}(x, a)}{\partial \log a}.$$

Recall that  $\widetilde{\mathcal{W}}_{T[M_K]}^{\text{eff}}(x, a)$  comes from integrating out the dynamical fields in the twisted superpotential (see (7)), which can be read from (25) or double-scaling limit (5) with  $N \rightarrow \infty$ ,  $q^N = a$ :

$$F_K(x, a, q) \xrightarrow{\hbar \rightarrow 0} \int \prod_i \frac{dz_i}{z_i} \exp \left( \frac{1}{\hbar} \widetilde{\mathcal{W}}_{T[M_K]}(z_i, x, a) + \mathcal{O}(\hbar^0) \right) \xrightarrow[\hbar \rightarrow 0]{r, N \rightarrow \infty} J_r^{\text{stN}}(K; q).$$

Note that the construction of  $A$ - and  $B$ -polynomials via the effective twisted superpotential (8, 73) immediately leads to the constraint

$$(74) \quad \frac{\partial \log y}{\partial \log a} = \frac{\partial^2 \widetilde{\mathcal{W}}_{T[M_K]}^{\text{eff}}(x, a)}{\partial \log x \partial \log a} = \frac{\partial \log b}{\partial \log x},$$

which is equivalent to the holomorphic Lagrangian condition:

$$\Omega = d \log x \wedge d \log y + d \log a \wedge d \log b = \frac{\partial \log y}{\partial \log a} d \log x \wedge d \log a + \frac{\partial \log b}{\partial \log x} d \log a \wedge d \log x = 0.$$

On the other hand, it allows us to derive  $A(x, y, a)$  from  $B(a, b, x)$  up to a function  $f(x)$ . Namely, we can solve  $B(a, b, x) = 0$  for  $b(a, x)$ , integrate over  $\log a$

$$\widetilde{\mathcal{W}}_{T[M_K]}^{\text{eff}}(x, a) = \int \log b(a, x) d(\log a) + f(x),$$

and differentiate with respect to  $\log x$ :

$$A_K(x, y, a) = 0 \quad \Leftrightarrow \quad \log y = \frac{\partial}{\partial \log x} \left( \int \log b(a, x) d(\log a) + f(x) \right).$$

We can apply the same reasoning to derive  $B(a, b, x)$  from  $A(x, y, a)$  up to a function  $f(a)$ .

Let us present the relations discussed above on the example of the unknot. In the fully unreduced normalisation we have

$$J_r^{\text{stN, fully unreduced}}(0_1; q) \xrightarrow[\hbar \rightarrow 0]{r, N \rightarrow \infty} \exp \left[ \frac{1}{\hbar} \left( -\frac{\log x \log a}{2} + \text{Li}_2(x) - \text{Li}_2(xa) + \text{Li}_2(a) - \frac{\pi^2}{6} \right) \right],$$

so the twisted superpotential is given by

$$\widetilde{\mathcal{W}}_{T[M_{0_1}]}(x, a) = -\frac{\log x \log a}{2} + \text{Li}_2(x) - \text{Li}_2(xa) + \text{Li}_2(a) - \frac{\pi^2}{6}.$$

This is consistent with [FGS13] as we do not have dynamical fields and  $\widetilde{\mathcal{W}}_{T[M_{0_1}]}(x, a) = \widetilde{\mathcal{W}}_{T[M_{0_1}]}^{\text{eff}}(x, a)$ . Plugging it in equation (73), we obtain

$$(75) \quad \log b = -\frac{\log x}{2} + \log(1 - xa) - \log(1 - a).$$

It leads to

$$(76) \quad B_{0_1}^{\text{fully unreduced}}(a, b, x) = (1 - a)b - x^{-1/2}(1 - xa) = 0,$$

which is in line with the classical limit of (71). One can also check that  $A_{0_1}(x, y, a) = (1 - x)y - a^{1/2}(1 - ax)$  can be obtained from (75-76) with  $f(x) = \text{Li}_2(x) - \frac{\pi^2}{6}$ .

Classical  $B$ -polynomials for simple knots in the reduced normalisation are given in Table 2.

$K$	$B_K(a, b, x)$
$0_1$	$1 - b$
$3_1$	$x^2 - x(2 - (1+x)a + x^2a^2)b + (1-a)(1-xa)b^2$
$4_1$	$x^2a^2 + ax(2 - (1+3x+x^2)a + 2x^2a^2)b$ $+ (1-a)(1-xa)(1-2x(1+x)a + x^3a^2)b^2$ $- x(1-a)(1-a)(1-xa)(1-xa)b^3$
$5_1$	$x^6 - x^4(3 - 2(1+x)a + x(1+2x)a^2 - x^2(1+x)a^3 + x^4a^4)b$ $+ x^2(1-a)(1-xa)(3 - (1+x)a + 2x^2a^2)b^2$ $- (1-a)(1-a)(1-xa)(1-xa)b^3$
$5_2$	$1 - x^{-2}(2a^2x^3 + a^2x^2 - 4ax^2 - ax - a + 3x + 1)b$ $- x^{-3}(a-1)(ax-1)(a^3x^4 - 3a^2x^3 - 2a^2x^2 + 5ax^2 + ax + a - 3x - 3)b^2$ $- x^{-4}(a-1)^2(ax-1)^2(a^2x^3 - 2ax^2 - ax + x + 3)b^3$ $+ x^{-5}(a-1)^3(ax-1)^3b^4$

TABLE 2. Classical  $B$ -polynomials for some simple knots.

**Remark 11.** If we set  $x = 1$ , we see that  $B_K(a, b, x = 1)$  always has a factor of  $b - 1$ . This is analogous to the presence of the factor  $y - 1$  in  $A_K(x, y, a = 1)$ . This condition lets us fix the integration constants coming from equation (74). It similarly allows us to reconstruct  $\Gamma_K$  from either  $A_K$  or  $B_K$ .

**Remark 12.** Thanks to the relation  $\frac{\partial \log b}{\partial \log x} = \frac{\partial \log y}{\partial \log a}$ , we can solve for  $y$  and vice versa for any branch of  $b$ . It follows that there is a canonical one-to-one correspondence between the branches of  $b$  and the branches of  $y$ , as functions of  $x$  and  $a$ . This correspondence will be used later in Section 6.5 to determine  $F_K$  for various branches, by solving the recurrence relation with respect to quantum  $A$ - and  $B$ -polynomials at the same time.

**Remark 13.** Practically, there are many ways to compute the classical  $B$ -polynomial. One way is to start from a quiver expression, eliminate variables from the quiver  $A$ -polynomials to get the ideal defining the holomorphic Lagrangian  $\Gamma_K$ , and then eliminate variable  $y$  to project it down to get the  $B$ -polynomial. Another way is to start from  $F_K$  associated to various branches and compute the expectation values of the  $\hat{b}$  operator. One gets  $b^{(\alpha)}(x, a)$  in a power series form. The elementary symmetric functions of  $b^{(\alpha)}$ 's are Laurent polynomials in  $x$  and  $a$ , and they are the coefficients of the  $B$ -polynomial.

6.2.4. *Relations to Alexander and HOMFLY-PT polynomials.* Recall from [DE20, EGG+20] that in case of the abelian branch

$$\lim_{q \rightarrow 1} \frac{F_K(x, qa, q)}{F_K(x, a, q)} = b(x, a) = \exp\left(\frac{\partial \widetilde{\mathcal{W}}(x, a)}{\partial \log a}\right) = \exp\left(\frac{\partial U_K(x, a)}{\partial \log a}\right)$$

is an  $a$ -deformation of the inverse of the Alexander polynomial, where  $U_K(x, a)$  denotes the Gromov-Witten disk potential for the knot complement. In particular, when  $a = 1$ ,  $b = \frac{1}{\Delta_K(x)}$ . This also means that

$$B_K(a = 1, \frac{1}{\Delta_K(x)}, x) = 0.$$

Knot	$\langle \hat{y} \rangle _{(x,b)=(1,1)}$
$\mathbf{3}_1^l$	$2a - a^2$
$\mathbf{3}_1^r$	$-a^{-2} + 2a^{-1}$
$\mathbf{4}_1$	$a^{-1} - 1 + a$
$\mathbf{5}_2$	$a + a^2 - a^3$
$\mathbf{6}_1$	$a^{-2} - a^{-1} + a$
$\mathbf{7}_2$	$a + a^3 - a^4$
$\mathbf{8}_1$	$a^{-3} - a^{-2} + a$

TABLE 3. Expectation value of  $\hat{y}$  at  $(x, b) = (1, 1)$ 

In case of non-abelian branches, it turns out  $b(x, a)$  has a pole at  $a = 1$ . As a result, the  $a = 1$  specialization of the classical  $B$ -polynomial is simply

$$B_K(a = 1, b, x) = 1 - \Delta_K(x)b$$

up to an overall multiplication by a monomial.

The main theorem of [DE20] states that

$$\langle \hat{b} \rangle = \exp\left(\int \frac{\partial \log y}{\partial \log a} d \log x\right) = \exp\left(-\int \frac{\partial_{\log a} A_K}{\partial_{\log y} A_K} d \log x\right)$$

becomes  $\frac{1}{\Delta_K(x)}$  when  $(a, y) = (1, 1)$ . Inspired by this, we can consider the “ $B$ -polynomial analogue” of this theorem:

$$\langle \hat{y} \rangle = \exp\left(\int \frac{\partial \log b}{\partial \log x} d \log a\right) = \exp\left(-\int \frac{\partial_{\log x} B_K}{\partial_{\log b} B_K} d \log a\right),$$

and when  $(x, b) = (1, 1)$ , it should give us the “ $B$ -polynomial analogue of  $\frac{1}{\Delta_K(x)}$ ”. These can also be directly obtained by solving the equation  $A_K(x = 1, y, a) = 0$  for  $y$ . It turns out that these are just HOMFLY-PT polynomials  $P(K; a, q = 1)$  specialized to  $q = 1$ . In fact, it is a well-known property of (reduced) coloured HOMFLY-PT polynomials that  $P_n(K; a, q = 1) = P_1(K; a, q = 1)^n$ , so it is consistent with what we have just observed!

**6.2.5. Geometric properties of the  $B$ -polynomial.** We discuss geometric aspects of the  $B$ -polynomial ranging from the known to the more conjectural. Consider the Legendrian conormal torus of a knot in the unit cotangent bundle of  $S^3$  filled by the conormal or complement Lagrangian in the resolved conifold. Let  $U_K(x, a)$  denote its disk potential. Consider a braid representation of  $K$  and take the limit where  $K$  collapses onto a multiple of the unknot. In the resolved conifold the corresponding multicover of the toric brane at the vertex arises as the limit of the complement and conormal Lagrangians, and the  $x$ -cycle and  $y$ -cycle are restored when this Lagrangian is shifted along different legs. Holomorphic curves converge in the corresponding limit to basic curves on the corresponding toric brane and we find that for the knot complement such curves are combinations of basic disks in homology classes  $x$  and  $xa$ . It follows that holomorphic curves on the knot complement have homology class  $x^k(xa)^l$ , where  $k \geq 0$  and  $l \geq 0$ . In particular, the disk potential can be expressed as

$$U_K(x, a) = \sum_{k,l \geq 0} c_{k,l} x^{k+l} a^l.$$

We first consider the geometric interpretation of  $\partial_{\log x} \partial_{\log a} U_K$ . The action of the  $\partial_{\log x}$ -operator can be understood in the following way: fix a cycle in  $M_K$  filling the longitude and add a boundary marked point on the disks at intersection points with this cycle. On the other hand, the action of  $\partial_{\log a}$ -operator means: fix a fiber in the resolved conifold dual to  $\mathbb{CP}^1$  and add an interior marked point at intersections with this fiber.

Consider now adding the fiber near the other vertex in the toric diagram. Inserting a toric brane  $L$  there, it is easy to see that we can open up any intersection to a small boundary on this toric brane. This means that  $\partial_{\log a} \partial_{\log x} U_K(x, a)$  counts holomorphic annuli stretching between  $M_K$  and  $L$  with a boundary marked point on both boundary components. To state this more precisely, let  $\log \xi$  and  $\log \eta$  be a homology basis for the torus at the boundary of  $L$  and write  $V_K(x, \xi, a)$  for the count of annuli stretching between  $L$  and  $M_K$ . Then

$$\partial_{\log x} \partial_{\log a} U_K(x, a) = \partial_{\log x} \partial_{\log \xi} V_K(x, \xi = a, a),$$

which shows that the variable  $\eta$  dual to  $\xi$  is related to  $b$  dual to  $a$ , but generally not equal to it, see Figure 9.

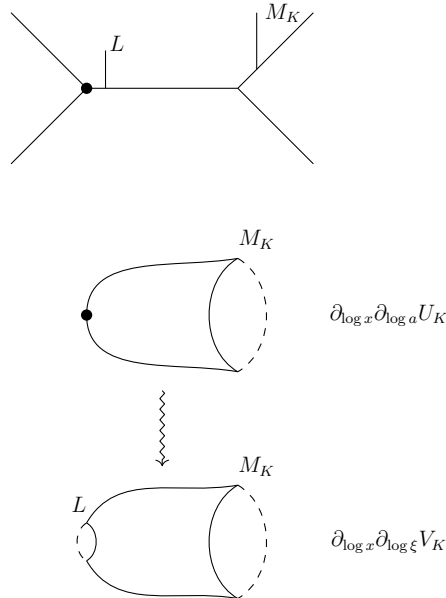


FIGURE 9. Opening up a disk intersection to annulus

Note next that from the knot theory perspective, the boundary of the toric brane  $L$  is the conormal Legendrian of a braid axis of  $K$ . This means that the annulus count  $V_K(x, a)$  can be computed from (the partial information about) the dg-algebra of the link  $K \cup S$ , where  $S$  is the braid axis. More precisely, using one-dimensional curves with a positive puncture at the degree 1 Reeb chord of  $L$ , we find the equation

$$(77) \quad \Delta_K + \partial_{\log \eta} A_L \cdot \partial_{\log x} \partial_{\log \xi} V_K(x, a) = 0,$$

where  $\Delta_K$  counts holomorphic triangles filled by disks with two positive punctures, see Figure 10.

Note also that  $L$  is simply the conormal of an unknot and therefore its augmentation polynomial  $A_L$  is well-known. When  $K$  is the unknot, this calculation is a calculation for the Hopf link that was carried out in [EN18]. In this case, because of the simplicity of

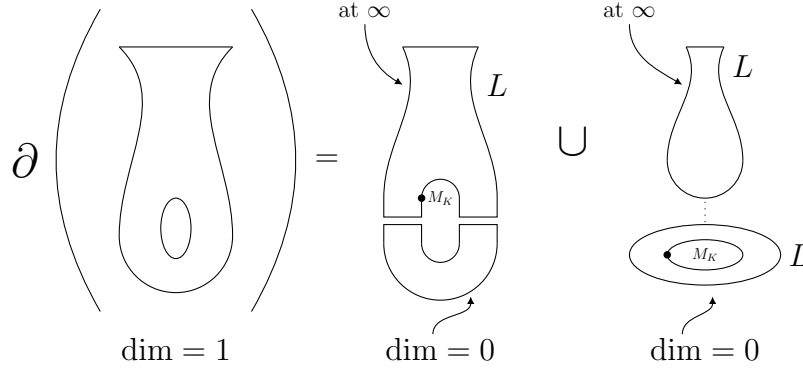


FIGURE 10. The boundary of the 1-dimensional moduli space of annuli stretching between  $L$  and  $M_K$  with positive puncture at one degree one Reeb chord of  $L$

the curves involved,  $V_U(x, \xi, a)$  is independent of  $a$  and the  $B$ -polynomial of the unknot can be computed directly from the above equation.

In general, by the definition of  $b$  and differentiation, the  $B$ -polynomial satisfies a similar equation:

$$(78) \quad \partial_{\log x} B_K + \partial_{\log \eta} B_K \cdot \partial_{\log x} \partial_{\log \xi} U_K(x, a) = 0.$$

Geometrically, the difference between (77) and (78) is that in the former there are no annuli at infinity – all annuli come from disks with marked points.

We next speculate what kind of theory would allow us to identify  $\xi = a$  and  $\eta = b$  and give a direct geometric interpretation of  $b$ . Consider the configuration of  $M_K$  and  $L$  in the resolved conifold as above. As  $a \rightarrow \infty$ , any holomorphic disk on  $L_K$  with  $k$  marked points at the point where  $L$  is inserted contains in the limit a disk component where all the marked points collide, with a sphere with  $k$  marked points on it attached. Consider now  $S^1$ -equivariant curves over the split off  $\mathbb{C}\mathbb{P}^1$ . Then any sphere in class  $a^m$  has  $m$ -fold branch points at the vertices. From the point of view of the brane  $L$ , the split curve looks like a disk with a curve with some  $x$ -charge split off at the center. It is natural to conjecture that the splitting gives rise to a new contact form at infinity (on  $S^3 \times S^2$  with a defect determined by the Legendrian conormal of  $K$ ) and that for this contact manifold the open string theory for  $L$  is related to the string theory of  $L_K$  simply by  $\xi = a$  and  $\eta = b$ , and the  $B$ -polynomial would arise simply as the augmentation polynomial of the dg-algebra of the Legendrian torus at infinity.

In the previous section we observed two properties of the polynomials  $B_K(a, b, x = 1)$ : first, the coefficient of the top power of  $b$  is divisible by  $(1 - a)$  and second, the polynomial contains the factor  $(1 - b)$ . Assuming that the theory discussed above exists, these properties of the  $B$ -polynomial follow from usual arguments. The  $x = 1$  limit corresponds to no shifting, no initial annuli and therefore no disk potential, and  $b = 1$  is a solution for which the  $\eta$ -cycle in  $L$  contracts without correction (the flux through the cycle is zero).

To see that the coefficient of the top  $b$ -degree term must contain a factor  $a - 1$ , we first recall how a similar property of the  $A$ -polynomial is derived. First, the coefficient of the top power of  $y$  is divisible by  $(1 - x)$  and second, at  $a = 1$  the polynomial contains the factor  $(1 - x)(1 - y)$ . To see this, we use the geometric interpretation of the equation

$$A_K(x(y), y, a) = 0$$

as the count of ends of the moduli space of generalized holomorphic disks with boundary on  $L_K$  and one positive puncture. Here the boundary corresponds to disks on the Legendrian torus at infinity, where the solution corresponds to inserting rigid disks at intersections with bounding chains at infinity, whereas the polynomial  $A_K(x, y, a)$  counts augmented disks in the  $\mathbb{R}$ -invariant region. Consider a change of variables of the degree zero Reeb chords such that the boundary of all augmentation disks have homology class a non-negative multiple of  $\log y$ . After a similar change of variables for the degree one Reeb chords, we arrange that the minimum degree coefficient of augmented disks at infinity also equals zero. Consider now following a degree zero disk when it moves into the filling. It cannot split off any disk since any disk without positive puncture has boundary which is a positive multiple of  $\log y$  by positivity of area of holomorphic disks. Also, if the degree zero disk picks up a rigid disk, its total homology class turns positive. It follows that the coefficient of the constant term in the augmentation polynomial must have a solution corresponding to a cycle that shrinks without splitting, which means that it contains a factor  $(1 - x)$  or  $(1 - ax)$ , depending on the choice of capping disk for  $x$ .

Consider now  $L$  in the resolved conifold with defect, sitting at the vertex. Expressing the theory in terms of  $\xi$  or  $\eta$  corresponds to shifting  $L$  along different legs in the toric diagram. In the shift where we expand in  $\eta = b$ , the above argument for  $A_K$  gives the observed property for  $B_K$ .

### 6.3. The Quantum $AB$ -ideal.

6.3.1. *The quantum  $AB$ -ideal.* Given a knot  $K$ , consider the set of recursion relations

$$\widehat{AB}_K = \{\hat{T} \in \mathbb{C}[\hat{x}^{\pm 1}, \hat{a}^{\pm 1}, q^{\pm 1}, \hat{y}^{\pm 1}, \hat{b}^{\pm 1}] \mid \hat{T}F_k(x, a, q) = 0\}.$$

It is clear that  $\widehat{AB}_K$  is a left ideal and that

$$\widehat{AB}_K \cap \mathbb{C}[\hat{x}^{\pm 1}, \hat{a}^{\pm 1}, q^{\pm 1}, \hat{y}^{\pm 1}] = \langle \hat{A}_K \rangle, \quad \widehat{AB}_K \cap \mathbb{C}[\hat{x}^{\pm 1}, \hat{a}^{\pm 1}, q^{\pm 1}, \hat{b}^{\pm 1}] = \langle \hat{B}_K \rangle.$$

Hence  $\widehat{AB}_K$  is a generalisation of both the  $\hat{A}$  and  $\hat{B}$ -polynomials and indeed it is exactly the *quantum  $AB$ -ideal* we mentioned earlier. There is a natural classical limit of this picture via the ring homomorphism which sends  $q \rightarrow 1$  and maps the quantum  $AB$ -ideal to the classical  $AB$ -ideal. A question we can immediately ask is how much information is lost when moving to the classical ideal. The most we can hope for is captured in the following conjecture.

**Conjecture 8.** *Given a set of polynomials  $T_1, \dots, T_n$  which generate  $AB_K$ , there exist quantizations  $\hat{T}_1, \dots, \hat{T}_n$  which generate  $\widehat{AB}_K$ .*

The main obstacle in studying the  $AB$ -ideal compared to either the  $A$ - or  $B$ -polynomials is that it is much more difficult to compute. In most cases all we can say is that we have found a sub-ideal which we suspect is the whole ideal. With that being said, we are able to show that in general this ideal is richer than simply the  $A$ - and  $B$ -polynomials combined.

We call  $AB_K$  *simple* if  $AB_K = \langle A_K, B_K \rangle$ , and for the collection of knots we studied  $AB_K$  is simple only for the unknot. This is a slightly surprising result as recently a similar phenomenon was studied in [MM21], where authors worked with the equivalent of the  $A$ - and  $B$ -polynomials coming from coefficients of the cyclotomic expansion of the coloured Jones polynomial. In that case they claim that their version of the  $AB$ -ideal is always simple. Further work needs to be done to understand the origin of the difference between these results.

6.3.2. *Unknot.* For the unknot we work with the unreduced normalisation

$$F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q) = \frac{(xq; q)_\infty}{(xa; q)_\infty}$$

as the ideal is clearly uninteresting when working with the reduced normalisation<sup>17</sup>. In this case we can easily describe the action of the  $\hat{y}$  and  $\hat{b}$  operators:

$$\begin{aligned}\hat{y}F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q) &= \frac{1 - ax}{1 - qx} F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q), \\ \hat{b}F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q) &= (1 - ax) F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q).\end{aligned}$$

and this instantly gives us three linear elements which clearly generate the ideal:

$$\hat{b} - (1 - \hat{a}\hat{x}), \quad (1 - q\hat{x})\hat{y} - (1 - \hat{a}\hat{x}), \quad (1 - q\hat{x})\hat{y} - \hat{b}.$$

In this case the third element is the difference of the first two, so  $AB_{\mathbf{0}_1} = \langle A_{\mathbf{0}_1}, B_{\mathbf{0}_1} \rangle$  and we conclude that the  $AB$ -ideal of the unknot is simple.

6.3.3. *Trefoil.* For knots more complicated than the unknot, it is not easy to simply compute  $I_K$  from observation. Instead, we use the quiver forms introduced earlier. Given a knot  $K$ , let  $Q$  be an associated quiver with identifications  $x_i \mapsto x^{n_i} a^{a_i} q^{l_i}$ . Each node  $i$  gives rise to a classical quiver  $A$ -polynomial as in equation (27). Combining these polynomials together, we obtain the classical quiver ideal  $I_Q \subset \mathbb{C}[\mathbf{x}^{\pm 1}, \mathbf{y}]$ . The identifications

$$y = \prod_i y_i^{n_i}, \quad b = \prod_i y_i^{a_i}$$

embed  $\mathbb{C}[\mathbf{x}^{\pm 1}, y, b]$  as a subring inside  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$  and there is a map

$$\begin{aligned}j_K : \mathbb{C}[\mathbf{x}^{\pm 1}, y, b] &\rightarrow \mathbb{C}[x^{\pm 1}, a^{\pm 1}, y, b], \\ x_i &\mapsto x^{n_i} a^{a_i}.\end{aligned}$$

Hence we can map  $I_{Q_K}$  to an ideal in  $\mathbb{C}[x^{\pm 1}, a^{\pm 1}, y, b]$  by intersecting with  $\mathbb{C}[\mathbf{x}^{\pm 1}, y, b]$  and applying  $j_K$ . This is the classical  $AB$ -ideal<sup>18</sup>:

$$AB_K = j_K(I_{Q_K} \cap \mathbb{C}[\mathbf{x}, y, b]).$$

Given a small quiver, this computation can be performed using Gröbner basis, so in principle we can compute classical  $AB$ -ideals. However, it is still highly nontrivial to re-quantize  $AB_K$  to construct the quantum ideal. Currently the only approach is essentially trial and error. Additionally, it is not clear how to prove Conjecture 8, so the resultant ideal may not be the entire quantum ideal.

For a more concrete example, let us work with the trefoil. Recall from Section 4.2 that it has a quiver form with

$$C = \begin{pmatrix} 0 & -1 & 0 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \quad \begin{aligned} \mathbf{n} &= (2, 2, 1, 1), \\ \mathbf{a} &= (1, 0, 1, 0), \\ \mathbf{l} &= \frac{1}{2}(0, 3, -1, 1). \end{aligned}$$

<sup>17</sup>As  $F_{\mathbf{0}_1}(x, a, q) = 1$  in the reduced normalisation.

<sup>18</sup>In principle, the intersection can leave behind extra components. In this case the  $AB$ -ideal is the unique component whose projections to  $\mathbb{C}_{x,y,a}$  and  $\mathbb{C}_{a,b,x}$  are the  $A$ - and  $B$ -ideals.

From this setup we can write down the four classical quiver  $A$ -polynomials:

$$\begin{aligned} A_1(\mathbf{x}, \mathbf{y}) &= 1 - y_1 + x_1 y_2^{-1} y_4^{-1}, & A_2(\mathbf{x}, \mathbf{y}) &= 1 - y_2 + x_2 y_1^{-1} y_2^{-1} y_4^{-1}, \\ A_3(\mathbf{x}, \mathbf{y}) &= 1 - y_3 + x_3 y_3, & A_4(\mathbf{x}, \mathbf{y}) &= 1 - y_4 + x_4 y_1^{-1} y_2^{-1}, \end{aligned}$$

and see that our  $y$  and  $b$  operators are given by

$$y = y_1^2 y_2^2 y_3 y_4, \quad b = y_1 y_3.$$

We can use Mathematica to compute the intersection and then manually remove the spurious irreducible components. Passing to the ideal, we shift  $b$  and  $y$  to incorporate the the prefactor of  $x^{-1} e^{\frac{\log(x)\log(a)}{\log(q)}}$  and we are left with

$$\langle B_{\mathbf{3}_1}, 1 - x^{-1}(1 - a)(1 - ax^2)b - y \rangle.$$

For simplicity, we call the other polynomial  $D_{\mathbf{3}_1}$ .

**Proposition 5.** *We have a strict containment of ideals:*

$$\langle A_{\mathbf{3}_1}, B_{\mathbf{3}_1} \rangle \subsetneq \langle B_{\mathbf{3}_1}, D_{\mathbf{3}_1} \rangle.$$

*Proof.* Containment follows from the equality

$$(79) \quad A_{\mathbf{3}_1} = \frac{(1-a)(1-ax^2)^2}{a^2} B_{\mathbf{3}_1} - \frac{1}{a^2} \left( (1-a - 2ax^2 + 2a^2x^2 + a^2x^3 - a^3x^4) \right. \\ \left. - x^{-1}(1-a)(1-ax)(1-ax^2)b + (1-ax)y \right) D_{\mathbf{3}_1}.$$

Proving that this containment is strict follows from a easy commutative algebra exercise.  $\square$

There are two obvious question to ask here. First, can we quantize  $D_{\mathbf{3}_1}$ , and second, can we promote Equation (79) to a quantum version which relates  $\hat{A}_{\mathbf{3}_1}$ ,  $\hat{B}_{\mathbf{3}_1}$ , and  $\hat{D}_{\mathbf{3}_1}$ . Through trial and error, we find that the answer to both questions is positive. One possible pair of quantizations is

$$\begin{aligned} \hat{D}_{\mathbf{3}_1} &= 1 - x^{-1}(1 - q^{-1}\hat{a})(1 - q\hat{a}\hat{x}^2)\hat{b} - \hat{y} \\ \hat{A}_{\mathbf{3}_1} &= \frac{q(1 - q^{-1}\hat{a})(1 - q\hat{a}\hat{x}^2)(1 - q^2\hat{a}\hat{x}^2)(1 - q^3\hat{a}\hat{x}^2)}{\hat{a}^2} \hat{B}_{\mathbf{3}_1} \\ &\quad - \frac{1}{\hat{a}^2} \left( q^2(1 - q^{-1}\hat{a} - (q + q^2)\hat{a}\hat{x}^2 + (1 + q)\hat{a}^2\hat{x}^2 + q^2\hat{a}^2\hat{x}^3 - q^2\hat{a}^3\hat{x}^4)(1 - q^3\hat{a}\hat{x}^2) \right. \\ &\quad \left. - qx^{-1}(1 - q^{-1}\hat{a})(1 - q\hat{a}\hat{x})(1 - q\hat{a}\hat{x}^2)(1 - q^3\hat{a}\hat{x}^2)\hat{b} + q^2(1 - q\hat{a}\hat{x})(1 - q\hat{a}\hat{x}^2)\hat{y} \right) \hat{D}_{\mathbf{3}_1}. \end{aligned}$$

Note that there is an interesting uniqueness question here as there are many ways to quantize both Equation (79) and  $D_{\mathbf{3}_1}$ . In particular, as both  $\hat{B}_{\mathbf{3}_1}$  and  $\hat{D}_{\mathbf{3}_1}$  annihilate  $F_{\mathbf{3}_1}$ , the multiplication of Equation (79) by any  $q$  factors will produce an element of the quantum ideal whose classical limit is the classical  $A$ -polynomial. This gives rise to a large family of elements whose classical limits all correspond to  $A_{\mathbf{3}_1}$ . In this particular case, we can define  $\hat{A}_{\mathbf{3}_1}$  as the minimal element of the family which lies in  $\mathbb{C}[\hat{y}^{\pm 1}, \hat{x}^{\pm 1}, a^{\pm 1}, q^{\pm 1}]$ , but in general there will not be a uniquely defined quantization. For a simple example of this phenomenon, consider the element

$$x^{-1}(1-a)(1-x)b^2 - a^{-1}y - (1-x)b + a^{-1}x^{-1}by,$$

which lies in the classical ideal. This is quantized by the family

$$a^{-1}x^{-2}(1-a)(1-q^i+aq^{1+i}x-aq^2x^2)\hat{b}^2-q^{i+1}a^{-1}\hat{y}-a^{-1}x^{-1}(1-q^i+aq^{i+1}x-aq^{2+i}x^2)\hat{b}+a^{-1}x^{-1}\hat{b}\hat{y}$$

for any integer  $i$ . Additionally, note that while we have only shown

$$\widehat{AB}_{\mathbf{3}_1} \supset \langle \widehat{B}_{\mathbf{3}_1}, \widehat{D}_{\mathbf{3}_1} \rangle,$$

this does imply that  $\widehat{AB}_{\mathbf{3}_1}$  is larger than  $\langle \widehat{A}_{\mathbf{3}_1}, \widehat{B}_{\mathbf{3}_1} \rangle$ , which means that the  $AB$ -ideal for the trefoil is not simple.

**6.3.4. Other Torus Knots.** For knots whose associated quivers are larger, passing through the quantum quiver  $A$ -polynomials runs into computational problems. It is still possible though to find interesting elements of the  $AB$ -ideal and show that the ideal is larger than the one generated by the  $A$ - and  $B$ -polynomials. As we show here, in certain cases we can upgrade known recursion relations to elements of this ideal. In [Hik04], Hikami shows the existence of a non-homogeneous recursion relation for the coloured Jones polynomial of the torus knot  $T_{2,2p+1}$ :

$$J_r(T_{2,2p+1}; q) = q^{pr} \frac{1 - q^{2r+1}}{1 - q^{r+1}} - q^{(2p+1)r+p+1} \frac{1 - q^r}{1 - q^{r+1}} J_{r-1}(T_{2,2p+1}; q).$$

Promoting  $q^r = x$  and replacing the Jones polynomial by  $F_p = F_{T_{2,2p+1}}$ , this becomes

$$(80) \quad (1 - qx)F_p(x, q) = x^p(1 - qx^2) - q^{p+1}x^{2p+1}(1 - x)F_p(q^{-1}x, q).$$

Next, recall that for the  $a$ -deformed  $F_K$  we have the following pair of relations:

$$F_K(x, q^2, q) = F_K(x, q), \quad F_K(x, q, q) = 1.$$

Incorporating this into (80), we find that

$$(1 - qx)F_p(x, q^2, q) = x^p(1 - qx^2)F_p(x, q, q) - q^{p+1}x^{2p+1}(1 - x)F_p(q^{-1}x, q^2, q).$$

This looks exactly like the  $N = 2$ ,  $a = q^2$  limit of a more general recursion relation. Through a little trial and error, we can restore the  $a$  dependence to get

$$(1 - q^{-1}ax)F_p(x, a, q) = x^p(1 - q^{-1}ax^2)F_p(x, q^{-1}a, q) - q^{-p-1}a^{p+1}x^{2p+1}(1 - x)F_p(q^{-1}x, a, q),$$

so we have a general recursion relation for  $F_p$ :

$$(1 - q^{-1}\hat{a}\hat{x}) - x^p(1 - q^{-1}\hat{a}\hat{x}^2)\hat{b}^{-1} + q^{-p-1}a^{p+1}\hat{x}^{2p+1}(1 - x)\hat{y}^{-1}.$$

It is easy to show that the classical limit of this polynomial does not lie in  $\langle A_p, B_p \rangle$  for any torus knot  $T_{2,2p+1}$  so that  $AB$ -ideal is not simple in all these cases.

**6.4. Refinement.** Almost everything we have discussed so far can be generalized to the refined (i.e.  $t$ -deformed) setting. Solving the recursion given by the quantum super- $A$ -polynomial  $\widehat{A}_K(\hat{x}, \hat{y}, a, q, t)$ , we obtain wave functions  $F_K^{(\alpha)}(x, a, q, t)$ , one for each branch  $\alpha$  of the super- $A$ -polynomial, which are  $t$ -deformations of the wave functions  $F_K^{(\alpha)}(x, a, q)$  we have discussed in previous sections.

Just like the unrefined wave functions, these wave functions admit quiver expressions. In fact, in all the examples we have considered, the quiver for the refined wave function is the same as the quiver for the unrefined one; refinement only affects the knots-quivers change of variables which now involves the variable  $t$ .

Once we have a quiver expression for a wave function  $F_K^{(\alpha)}(x, a, q, t)$ , the left-ideal of  $q$ -difference operators in  $\hat{x}, \hat{y}, \hat{a}, \hat{b}, \hat{t}, \hat{u}$  (where  $u$  is a variable conjugate to  $t$ ) that annihilates

the wave function can be obtained from the quantum quiver  $A$ -polynomials for the quiver expression, through non-commutative elimination theory. This quantum ideal is the quantisation of the ideal defining the complex 3-dimensional holomorphic Lagrangian subvariety  $\Gamma_K$  in Conjecture 3.

The classical holomorphic Lagrangian subvariety  $\Gamma_K \subset (\mathbb{C}^*)^6$  can be described and computed more explicitly, by applying elimination theory (e.g. Gröbner basis) to the ideal defined by the quiver  $A$ -polynomials.

Empirically, we find that the Weyl symmetry of the (unrefined) holomorphic Lagrangian subvariety described in Section 6.1 can be lifted to the Weyl symmetry of the refined holomorphic Lagrangian, which is stated in part (1) of Conjecture 3. When projected down to  $(\mathbb{C}^*)^4$  parametrised by variables  $x, y, a, t$ , this version of Weyl symmetry was already noted in equation (8.5) of [GNS<sup>+</sup>16].

There are some interesting limits regarding the new variable  $u$  conjugate to  $t$ . For example, the expectation value of the  $\hat{u}$ -operator on  $F_K^{(\text{ab})}(x, a, q, t)$ , when  $a = 1, t = -1$  is

$$\langle \hat{u} \rangle|_{a=1, t=-1} = \frac{1}{\Delta_K(x)}.$$

One way to see this is by setting  $t = -q^N$  in the equation (108) of [EGG<sup>+</sup>20]. Another way to see this is from  $\mathfrak{sl}_1$  pairs. Recall that the Alexander polynomial arises from a count of annuli and that to get the expression we are interested in  $\exp\left(\frac{\partial U_K}{2\partial \log a}\right)$  at  $a = 1$ . We should think of a deformed (stretched) situation where all annuli are generalised annuli consisting of disks with a 4-chain intersection. Assume now that there is some quiver description where the annuli we count are the basic disks with 4-chain intersection. Here a quiver node with ‘charges’  $a^r q^s t^l x^k$  would contribute with  $\frac{1}{2}r(-1)^l x^k$  annuli. Here  $r$  comes from the number of 4-chain intersection, and  $(-1)^l$  comes from the spin structure orientation sign. The interpretation of  $t$  is the number of twists in the trivialization along the boundary, and  $x^k$  is just the homology class of the boundary. Assume now that nodes come in  $\mathfrak{sl}_1$  pairs corresponding to factors  $(1 + tq^{-2}a^2)$  in the super polynomial. We would then have two nodes  $a^r t^l x^k$  and  $a^{r+2} t^{l+1} x^k$  (suppressing  $q$  powers which play no role). Taking derivative with respect to  $\log a^2$  and setting  $a = 1, t = -1$  or taking derivative with respect to  $\log t$  and setting  $a = 1, t = -1$  gives the same total contribution to annuli:  $(-1)^{l+1} x^k$ . After an overall change of framing, also the contributions to the  $\log a^2$  and  $\log t$  derivatives from the surviving  $\mathfrak{sl}_1$  node agree.

**6.5. Applications to computing  $F_K$  on different branches.** In this section we show how to use the quantum  $B$ -polynomial to further analyse  $F_K$  on different branches as introduced in Section 3.

**6.5.1. General argument.** In Section 3, sometimes we computed  $F_K$  recursively, using the quantum  $A$ -polynomial. In general, this approach gives solutions up to multiplication by a function of  $a$  and  $q$ . While it is not possible to completely eliminate this uncertainty, using the quantum  $B$ -polynomial we can fully determine the  $a$ -dependence. Then in the event that we know  $F_K$  for any specialization  $a = q^N$ , we can determine the  $q$ -dependence as well. This method relies on the observation that Proposition 1 and Conjecture 5 apply equally well to the  $B$ -polynomial, as well as the following conjecture.

**Conjecture 9.** *Let  $e_A$  be a left edge of the Newton polygon of the  $A$ -polynomial with slope  $\frac{n_x}{n_y}$ . Then there exists a left edge  $e_B$  of the Newton polygon of the  $B$ -polynomial with slope  $\frac{n_a}{n_b}$ .*

and a Puiseux series in  $x, a, q$  satisfying

$$(81) \quad \hat{A}_K F_K^{(e_A, e_B)} = 0 \quad \text{and} \quad \hat{B}_K F_K^{(e_A, e_B)} = 0.$$

This series has the form

$$F_K^{(e_A, e_B)}(x, a, q) = \exp\left(\frac{p(\log x, \log a)}{\hbar}\right) \left(1 + \sum_{i, j \geq 1} f_{i, j}(q) x^{\frac{i}{n_A}} a^{\frac{j}{n_B}}\right),$$

where  $p$  is a polynomial of degree at most 2 determined by  $e_A$  and  $e_B$ , and each coefficient  $f_{i, j}$  is itself a Laurent series in  $q$  with integer coefficients.

Note that we could have started off with left edges of the Newton polygon of the  $B$ -polynomial. Since, after dealing with degeneracies, we get a bijection between the two sets of left edges, the end result would be the same. To illustrate this further, we will focus on the example of the  $4_1$  knot.

6.5.2.  $4_1$  knot. The Newton polygon for the  $A$ -polynomial was shown in Figure 1, whereas the Newton polygon for the  $B$ -polynomial is presented in Figure 11. Looking at them, we see that both have three left edges when counted with multiplicity.

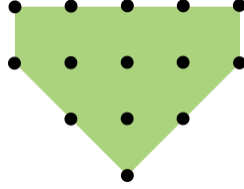


FIGURE 11. The Newton polygon for  $B_{4_1}$

The prefactors in Conjecture 5 arise from the slope of the edge on the Newton polygon and through the requirement that the  $F_K$  is computed in the form  $1 + O(x)$  or  $1 + O(a)$ . For the  $A$  polynomial, the prefactors are<sup>19</sup>

$$q^{r^2+rN+r}, \quad q^{rN-r}, \quad q^{-r^2-rN+R},$$

whereas for the  $B$ -polynomial they are given by

$$q^{-rN}, \quad e^{N\pi i} q^{\frac{N^2}{2}+rN-\frac{N}{2}}.$$

Note that one of the edges for the  $B$ -polynomial is degenerate, so the second prefactor appears with multiplicity 2. Next, we compute the  $F_K$  on various branches recursively. Starting with the  $A$ -polynomial, we find the three branches to be

$$\begin{aligned} F_{4_1}^{A, (ab)} &= q^{rN-r} \left( 1 - \frac{3(a-q)}{1-q} x - \frac{(a-q)(1-2a+6q-6aq-2q^2+aq^2)}{(1-q)(1-q^2)} x^2 + O(x^3) \right), \\ F_{4_1}^{A, (\frac{1}{2})} &= q^{-r^2-rN+r} \left( 1 + \frac{a(2-q)}{q(1-q)} x + \left( \frac{(1+3q-2q^2-2q^3+q^4)a^2}{q^3(1-q)(1-q^2)} - \frac{a}{q} \right) x^2 + O(x^3) \right), \\ F_{4_1}^{A, (-\frac{1}{2})} &= q^{r^2+rN+r} \left( 1 + \frac{q(1-2q)}{1-q} x + \left( \frac{q^2(1-2q-2q^2+3q^3+q^4)}{(1-q)(1-q^2)} - aq \right) x^2 + O(x^3) \right). \end{aligned}$$

<sup>19</sup>With  $r = \frac{\log(x)}{\hbar}$  and  $N = \frac{\log(a)}{\hbar}$ .

Similarly, the branches of  $B$ -polynomial are given by

$$\begin{aligned}
F_{\mathbf{4}_1}^{B,(\infty)} &= q^{-rN} \left( 1 - \frac{(1-x)(1-x+qx)}{q(1-q)} a \right. \\
&\quad \left. + \frac{(1-x)(q^2 - q(2-q^2)x + (1-q^2)(1+2q-q^2)x^2 - (1-q)(1-q^2)x^3)}{q^3(1-q)(1-q^2)} a^2 + O(a^3) \right), \\
F_{\mathbf{4}_1}^{B,(-1)} &= e^{N\pi i} q^{\frac{N^2}{2} + rN - \frac{N}{2}} \left( 1 + c_1(x, q) a \right. \\
&\quad \left. + \left( \frac{x(-3 + (1-q)^2x - q^2(2-q)x^2)}{(1-q)(1-q)^2} - \frac{1 + 3qx + q^2(1-q)x^2}{1-q^2} c_1 \right) a^2 + O(a^3) \right).
\end{aligned}$$

Note that  $c_1$  is an arbitrary function of  $x, q$  and appears due to the degeneracy of the  $-1$  slope edge. It remains to match up the  $F_K$ 's coming from the  $A$ - and  $B$ -polynomials. In this case, this can be done entirely from analysis of prefactors. As the  $A$ -recursion is defined up to factors of  $a, q$  and the  $B$ -recursion up to factors of  $x, q$ , we can pair up the branches and work out the overall prefactors. We obtain

$$\begin{aligned}
F_{\mathbf{4}_1}^{A,(ab)} &\sim F_{\mathbf{4}_1}^{B,(-1)} &\rightarrow e^{N\pi i} q^{\frac{N^2}{2} + rN - r - \frac{N}{2}}, \\
F_{\mathbf{4}_1}^{A,(\frac{1}{2})} &\sim F_{\mathbf{4}_1}^{B,(\infty)} &\rightarrow q^{-r^2 - rN + r}, \\
F_{\mathbf{4}_1}^{A,(-\frac{1}{2})} &\sim F_{\mathbf{4}_1}^{B,(-1)} &\rightarrow e^{N\pi i} q^{\frac{N^2}{2} + r^2 + rN + r - \frac{N}{2}}.
\end{aligned}$$

Once we have matched the series up, we can determine the combined series as we expect (ignoring prefactors)

$$F^{\text{combined}}(x, a, q) = F^A(x, a, q)F^B(0, a, q) = F^A(x, 0, q)F^B(x, a, q).$$

It turns out that in several cases computing the combined  $F_K$  is even easier. In particular, after correcting the prefactor  $F_{\mathbf{4}_1}^{A,(ab)}$  is annihilated by the  $B$ -polynomial and  $F_{\mathbf{4}_1}^{B,(\infty)}$  is annihilated by the  $A$  polynomial. This leaves the final pairing of  $F_{\mathbf{4}_1}^{A,(-\frac{1}{2})}$  and  $F_{\mathbf{4}_1}^{B,(-1)}$ , which is more difficult as we have the unknown constant  $c_1$ . Through some trial and error we find that the right value for  $c_1$  is  $1 + O(x)$  and

$$F_{\mathbf{4}_1}^{B,(\infty)}(0, a, q) = F_{\mathbf{4}_1}^{B,(-1)}(c_1, 0, a, q) \Big|_{c_1 = \frac{-1}{q(1-q)}} = \frac{1}{(aq^{-2}, q^{-1})_\infty} = (q^{-1}a; q)_\infty.$$

Hence the three branches of  $F_{4_1}$  are given by (up to an overall function of  $q$ )

$$\begin{aligned}
F_{4_1}^{(ab)} &= e^{N\pi i} q^{\frac{N^2}{2} + rN - r - \frac{N}{2}} \\
&\quad \times \left( 1 - \frac{3(a-q)}{1-q}x - \frac{(a-q)(1-2a+6q-6aq-2q^2+aq^2)}{(1-q)(1-q^2)}x^2 + O(x^3) \right), \\
F_{4_1}^{(\frac{1}{2})} &= q^{-R^2 - RN + R} (q^{-1}a; q)_\infty \\
&\quad \times \left( 1 + \frac{a(2-q)}{q(1-q)}x + \left( \frac{(1+3q-2q^2-2q^3+q^4)a^2}{q^3(1-q)(1-q^2)} - \frac{a}{q} \right) x^2 + O(x^3) \right), \\
F_{4_1}^{(-\frac{1}{2})} &= e^{N\pi i} q^{\frac{N^2}{2} + r^2 + rN + r - \frac{N}{2}} (q^{-1}a; q)_\infty \\
&\quad \times \left( 1 + \frac{q(1-2q)}{1-q}x + \left( \frac{q^2(1-2q-2q^2+3q^3+q^4)}{(1-q)(1-q^2)} - aq \right) x^2 + O(x^3) \right).
\end{aligned}$$

Observe that both non abelian branches which correspond to the slopes  $\frac{1}{2}$  and  $-\frac{1}{2}$  have poles in the  $q \rightarrow 1$  limit and disappear when  $a = q$ . On the other hand, the abelian branch is well-defined as  $q \rightarrow 1$  and in the  $a = q$  limit it is equal to 1.

**6.6. Closed sector recursions.** When we consider the generating functions of open topological strings, we usually mode out the whole closed sector. However, using  $B$ -polynomials and  $AB$ -ideals we can include it in the recursions.

6.6.1. *Closed partition function.* The closed sector partition function reads

$$\phi(a, q) = \exp \left[ \sum_d \frac{1}{d} \frac{a^d}{(1-q^d)^2} \right].$$

Since

$$\frac{\phi(a, q)}{\phi(aq, q)} = \psi(a, q) = \sum_d \frac{a^d}{(q)_d}, \quad (1 - \hat{a} - \hat{b}) \psi(a, q) = 0,$$

we can write

$$(1 - \hat{a}) \frac{\phi(a, q)}{\phi(aq, q)} - \frac{\phi(aq, q)}{\phi(aq^2, q)} = 0.$$

Therefore we have

$$(1 - \hat{a}) \phi(a, q) \hat{b}^2 \phi(a, q) - (\hat{b} \phi(a, q))^2 = 0.$$

Let us discuss the geometry underlying these formulas. Consider first the toric brane  $L$  in  $\mathbb{C}^3$ . The open string partition function

$$\psi(x, q) = \exp \left( \sum_d \frac{1}{d} \frac{x^d}{(1-q^d)} \right)$$

can be interpreted, after SFT-stretching around  $L$ , as the Gromov-Witten curve count. The coefficient of  $x^d$  counts the contribution from connected curves that are asymptotic to the multiplicity  $d$  Reeb orbit over the unique index zero geodesic in  $L$ . Consider now shifting  $x$  to  $qx$ . This can be realized geometrically by shifting the brane. We can then compute the new partition function  $\psi(xq, q)$  by stretching around both the original  $L_0$  and the shifted  $L_1$  branes. Denoting the  $d$ -fold Reeb orbits of  $L_j$  as  $\gamma_j^{(\pm d)}$ , there are basic cylinders with two

positive punctures inside the neighbourhood of  $L_0$  and a basic cylinder stretching between  $L_0$  and  $L_1$ . Then the corresponding  $d$ -fold covers contribute respectively as

$$\frac{1}{d}\gamma_0^{(d)}\gamma_0^{(-d)} \quad \text{and} \quad \frac{q^d}{d}\gamma_0^{(-d)}\gamma_1^{(d)}$$

to the count of connected curves, where  $dg_s = d\log(q)$  is the area of the cylinder stretching between the Lagrangians. (Here the two positive puncture cylinder has area 0 since it lies in the negative end. Its actual non-zero area is visible only after rescaling and taking any non-zero area to infinity.) Gluing these curves we find that

$$\psi(xq, q) = \exp\left(\sum_d \frac{d^2}{d^3} x^d q^d\right) = \exp\left(\sum_d \frac{1}{d} x^d q^d\right),$$

as expected. Here the  $d^2$ -factor comes from gluing along  $d$ -fold Reeb orbits twice. We point out that this gives a curve counting proof of the recursion relation

$$(1 - \hat{x} - \hat{y})\psi(x, q) = 0.$$

We will next give a similar curve counting proof of the closed string recursion. Consider local  $\mathbb{CP}^1$ . We first note that we can compute the closed string partition function by inserting a toric brane over the equator and applying SFT stretching. Connected curves glue over the area zero cylinders in the negative end and we find that

$$\phi(a^2) = \exp\left(\sum_d \frac{d^2}{d^3} \left(\frac{a^d}{(1-q^d)}\right)^2\right) = \exp\left(\sum_d \frac{1}{d} \frac{a^{2d}}{(1-q^d)^2}\right).$$

In order to find the effect of  $a \rightarrow aq$ , we argue as in the open case (or simply give area  $g_s$  to the cylinder in the negative end). The result is

$$\phi(qa^2) = \exp\left(\sum_d \frac{d^4}{d^5} q^d \left(\frac{a^d}{(1-q^d)}\right)^2\right),$$

where the holomorphic building consists of two outside curves, two cylinders in the negative ends and one cylinder stretching between Lagrangians. The curves are glued over multiple  $d$  Reeb orbits in four places. We conclude that

$$\phi(qa^2)\phi(a^2) = \exp\left(\sum_d \frac{1}{d}(1-q^d) \left(\frac{a^d}{(1-q^d)}\right)^2\right) = \psi(a^2),$$

where the factor of  $(1-q^d)$  comes from the area difference between a zero area cylinder in the negative end and a cylinder which is a  $d$ -fold cover of the of the basic area  $g_s$  cylinder that increases the area of the  $\mathbb{CP}^1$ .

6.6.2. *Combination with the unknot.* In the next step we combine the closed sector with the open one for the unknot (in the unreduced normalisation). Multiplying  $\phi$  by  $F_{\mathbf{0}_1}^{\text{unreduced}}$ , we obtain

$$\Phi(x, a, q) = \phi(a, q)F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q) = \exp\left[\sum_d \frac{1}{d} \frac{a^d}{(1-q^d)^2}\right] (xq; q)_{\frac{\log a}{h} - 1}.$$

Since

$$\hat{b}F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q) = (1 - x\hat{a})F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q),$$

we have

$$\frac{\Phi(x, a, q)}{\Phi(x, aq, q)} = \frac{\phi(a, q)}{\phi(aq, q)} \frac{F_{\mathbf{0}_1}^{\text{unreduced}}(x, a, q)}{F_{\mathbf{0}_1}^{\text{unreduced}}(x, aq, q)} = \frac{\psi(a, q)}{(1 - xa)}.$$

Therefore

$$(1 - \hat{a} - \hat{b})(1 - xa) \frac{\Phi(x, a, q)}{\Phi(x, aq, q)} = 0,$$

which leads to

$$(1 - a)(1 - xa)\Phi(x, a, q)\hat{b}^2\Phi(x, a, q) - (1 - xaq) \left( \hat{b}\Phi(x, a, q) \right)^2 = 0.$$

6.6.3. *Non-linear recursions.* One may wonder what happens if we consider  $\Phi_K = \phi F_K$  for more complicated knots  $K$ . It turns out that similar relations exist for all knots  $K$  and can be computed directly from the  $B$ -polynomial.

Observe that the quadratic relation for  $\phi$  actually follows from a  $B$ -polynomial whose coefficients involve the infinite  $q$ -Pochhammers. Namely,  $\phi$  satisfies the linear relation

$$\frac{\phi(qa, q)}{(a; q)_\infty} - \phi(a, q) = 0,$$

which means it is annihilated by  $B$ -polynomial  $\frac{1}{(\hat{a}; q)_\infty} \hat{b} - 1 = \hat{b}' - 1$ , where  $\hat{b}' = \frac{1}{(\hat{a}; q)_\infty} \hat{b}$ .

**Proposition 6.** *Given a knot  $K$  with  $B$ -polynomial*

$$B_K(\hat{b}, \hat{a}, x, q) = \sum_{i=0}^n c_i(\hat{a}, x, q) \hat{b}^i$$

*annihilating  $F_K$ ,  $\Phi_K$  is annihilated by  $B_K(\hat{b}', \hat{a}, x, q)$ , where  $\hat{b}'$  is defined above.*

The proof of this is immediate from the observation that  $\hat{b}'\Phi_K = \phi\hat{b}F_k$ . The remaining question is how to pass from this to the non-linear relations which do not involve infinite  $q$ -Pochhammers. To make notation a little easier, let  $\Phi_{K,n} = \hat{b}^n \Phi_K \hat{b}^{-n} = \Phi_K(q^n a, x, q)$  and observe that

$$(\hat{b}')^n = \left( \prod_{i=1}^n \frac{(a; q)_{i-1}}{(a; q)_\infty} \right) \hat{b}^n = \frac{\prod_{j=1}^n (a; q)_{j-1}}{(a; q)_\infty^n} \hat{b}^n.$$

Then the proposition above is equivalent to the statement

$$S[0] := \sum_{i=0}^n \left( \left( c_i(a, x, q) \left( \prod_{j=1}^i (a; q)_{j-1} \right) \Phi_{K,i} \right) \frac{1}{(a; q)_\infty^i} \right) = 0.$$

From this we can define  $S[j]$  by

$$S[k] := \frac{1}{((a; q)_k)^n} \hat{b}^k S[0] = \sum_{i=0}^n \left( \left( \frac{c_i(q^k a, x, q) \left( \prod_{j=1}^i (q^k a; q)_{j-1} \right)}{((a; q)_k)^{n-i}} \Phi_{K,j+i} \right) \frac{1}{(a; q)_\infty^i} \right) = 0.$$

Now consider the set of  $n+1$  equations  $S[0], \dots, S[n]$  as linear equations in the  $n+1$  variables  $1, \frac{1}{(a; q)_\infty}, \dots, \frac{1}{(a; q)_\infty^n}$ . Elementary linear algebra tells us that we can eliminate the variables  $\frac{1}{(a; q)_\infty^i}$  from the set of equations  $S[0], \dots, S[n]$  which will leave us with an equation purely in terms of  $\Phi_{K,i}, a, x, q$ . On top of this, the final equation will be a homogeneous polynomial of order  $n$  in the  $\Phi_{K,i}$ 's – exactly the nonlinear equation we are looking for.

The basic closed sector case corresponds to  $F_K = 1$ ,  $B_K = 1 - b$ , and if we plug this in, we directly recover

$$(1 - a)\phi(a, q)\phi(q^2a, q) - \phi(qa, q)^2 = 0.$$

If we set  $K = \mathbf{0}_1$  to be the unknot, we similarly recover the earlier formula

$$(1 - a)(1 - xa)\Phi_{\mathbf{0}_1,0}\Phi_{\mathbf{0}_1,2} - (1 - qxa)\Phi_{\mathbf{0}_1,1}^2 = 0.$$

Finally, setting  $K = \mathbf{3}_1$  to be the trefoil, we compute

$$\begin{aligned} 0 &= (a - 1)(aq^2 - 1)^2(aq - 1)^2(aq^3x - 1)(a^2q^3x^2 - aq^2x - aq + q + 1)\Phi_{\mathbf{3}_1,0}\Phi_{\mathbf{3}_1,2}\Phi_{\mathbf{3}_1,4} \\ &\quad + (a - 1)(aq - 1)^3(aq^2x - 1)(a^2q^5x^2 - aq^3x - aq^2 + q + 1)\Phi_{\mathbf{3}_1,0}\Phi_{\mathbf{3}_1,3}^2 \\ &\quad + (aq - 1)^2(aq^2 - 1)^2(aq^3x - 1)(a^2q^2x^2 - aqx - a + q + 1)\Phi_{\mathbf{3}_1,1}^2\Phi_{\mathbf{3}_1,4} \\ &\quad - (aq - 1)\left(a^4q^6x^3 - a^3q^6x^3 - a^3q^5x^2 + a^2q^5x^2 + a^4q^4x^3 - 2a^3q^4x^2 + a^2q^4x^2 - a^3q^3x^3 \right. \\ &\quad \quad \left. + a^2q^3x^2 - a^3q^2x^2 + a^2q^4x - 2a^3q^3x + 3a^2q^3x + 3a^2q^2x + a^2q^2 + a^2qx^2 \right. \\ &\quad \quad \left. + a^2qx + a^2q - 2aq^3x - 2aq^2x - 2aq^2 - 2aqx - 2aq - 2a + 2q + 2\right)\Phi_{\mathbf{3}_1,1}\Phi_{\mathbf{3}_1,2}\Phi_{\mathbf{3}_1,3} \\ &\quad - (a - 1)\Phi_{\mathbf{3}_1,2}^3(aqx - 1)(a^2q^3x^2 - aq^2x - aq + q + 1). \end{aligned}$$

Combining above discussion with results from Section 6.3, we can define open-closed  $\widehat{AB}$ -ideals by a redefinition of operator  $\hat{b}$ :

$$\hat{b} \rightarrow \hat{b}' = \hat{b}(\hat{a}; q)_\infty^{-1}.$$

For the unknot in the unreduced normalisation, it gives

$$\begin{aligned} \hat{A}_{\mathbf{0}_1}(\hat{x}, \hat{y}, a, q) &= (1 - \hat{x}q)\hat{y} - (1 - \hat{x}a) \rightarrow \hat{A}_{\mathbf{0}_1}^{\text{open-closed}}(\hat{x}, \hat{y}, a, q) = (1 - \hat{x}q)\hat{y} - (1 - \hat{x}a), \\ \hat{B}_{\mathbf{0}_1}(\hat{a}, \hat{b}, x, q) &= \hat{b} - (1 - x\hat{a}) \rightarrow \hat{B}_{\mathbf{0}_1}^{\text{open-closed}}(\hat{a}, \hat{b}, x, q) = \hat{b}(\hat{a}; q)_\infty^{-1} - (1 - x\hat{a}), \end{aligned}$$

and we have  $\widehat{AB}_{\mathbf{0}_1}^{\text{open-closed}} = \langle \hat{A}_{\mathbf{0}_1}^{\text{open-closed}}, \hat{B}_{\mathbf{0}_1}^{\text{open-closed}} \rangle$ .

## 7. CLOSED 3-MANIFOLDS AND LOG-CFT STRUCTURES

Authors of [GM21] proposed the following surgery formula connecting  $F_K$  with  $\widehat{Z}$  invariant for 3-manifold obtained by Dehn surgery in which we glue the complement of  $K$  and  $-1/r$  solid torus:

$$(82) \quad \widehat{Z}(S_{-1/r}^3(K)) = \sum_{n=1}^{\infty} q^{\frac{(nr-1)^2}{4r}} (q^n - 1) f_n(q),$$

where

$$\sum_n f_n(q) \cdot x^{\frac{n}{2}} = -x(x^{\frac{1}{2}} - x^{-\frac{1}{2}})F_K(q^{-1}x, q),$$

which comes from matching the conventions used in this work and in [GM21].

If we write  $F_K(x, q) = \sum_{n \geq 0} \tilde{f}_n(q) \cdot x^n$ , then (82) leads to

$$\widehat{Z}(S_{-1/r}^3(K)) = - \sum_{n \geq 0} \tilde{f}_n(q) \left[ q - q^{2r+2nr} - q^{2+2n} + q^{3+2n+2r+2nr} \right] q^{rn^2+rn-2n+\frac{r}{4}+\frac{1}{4r}-\frac{3}{2}}.$$

For each  $F_K$  that can be expressed in a quiver form<sup>20</sup>

$$(83) \quad F_K(x, q) = \sum_{\mathbf{d} \geq 0} q^{\frac{1}{2} \mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}} \frac{\mathbf{x}^{\mathbf{d}}}{(q)_{\mathbf{d}}}$$

with  $x_i \propto x$ , we have  $n = \sum_i d_i$  and, therefore,  $\widehat{Z}(S^3_{-1/r}(K))$  is a linear combination of “characters”

$$-\sum_{\mathbf{d} \geq 0} \frac{1}{(q)_{\mathbf{d}}} q^{r(\sum_i d_i)^2 + \frac{1}{2} \mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d} + (\text{terms linear in } \mathbf{d})} = -\sum_{\mathbf{d} \geq 0} \frac{1}{(q)_{\mathbf{d}}} q^{\frac{1}{2} \mathbf{d} \cdot \mathbf{C}^{(2r)} \cdot \mathbf{d} + (\text{terms linear in } \mathbf{d})}.$$

Effectively, this means that the modification of matrix  $C$  is the same as framing by  $2r$  (see [KRSS19]). For example, for the right-hand trefoil  $\widehat{Z}(S^3_{-1/r}(K))$  is a linear combination of Nahm sums with the matrix

$$(84) \quad C^{(2r)} = \begin{pmatrix} 2r & 2r+1 & 2r & 2r \\ 2r+1 & 2r & 2r+1 & 2r \\ 2r & 2r+1 & 2r+1 & 2r \\ 2r & 2r & 2r & 2r+1 \end{pmatrix},$$

whereas for the figure-eight we have the matrix

$$C^{(2r)} = \begin{pmatrix} 2r & 2r-1 & 2r & 2r & 2r-1 & 2r \\ 2r-1 & 2r & 2r & 2r & 2r & 2r+1 \\ 2r & 2r & 2r & 2r & 2r & 2r \\ 2r & 2r & 2r & 2r+1 & 2r & 2r \\ 2r-1 & 2r & 2r & 2r & 2r+1 & 2r+1 \\ 2r & 2r+1 & 2r & 2r & 2r+1 & 2r+1 \end{pmatrix}.$$

We expect the corresponding element of the Bloch group and the value of  $c_{\text{eff}}$  to be qualitatively different for  $r = 1$  compared to  $r > 1$ .

**7.1. Anomalies.** Recalling the analysis from Section 2.4.3, we can derive the field content and interactions of the 3d  $\mathcal{N} = 2$  theory  $T[Q]$  and dual theory  $T[M_K]$ . The matrix  $C$  is the matrix of Chern-Simons coefficients for  $U(1)^m$  gauge theory. Indeed, we write

$$q = e^{\hbar}, \quad y_i = q^{d_i} = e^{\hbar d_i}$$

and, as usual, take the double-scaling limit  $\hbar \rightarrow 0$  while keeping  $\hbar d_i$  fixed. In this limit, using  $d_i = \frac{1}{\hbar} \log y_i$ , from the quadratic term  $\frac{1}{2} \mathbf{d} \cdot \mathbf{C} \cdot \mathbf{d}$  we get

$$(85) \quad \widetilde{\mathcal{W}}_K = \frac{1}{2} \sum_{ij} C_{ij} \log y_i \log y_j + \dots$$

Note, the powers of  $q$  linear in  $\mathbf{d}$  do not contribute to the twisted superpotential. We denote it by  $\widetilde{\mathcal{W}}_K$  to stress that we assume the knots-quivers identification  $x_i = (-1)^{t_i} q^{l_i} a^{a_i} x^{n_i}$ , which implies that the moduli space of supersymmetric vacua of theories  $T[Q]$  and  $T[M_K]$  is the same, by construction.

<sup>20</sup>Note that for convenience in this section we moved  $(-1)^{C_{ii} d_i^2} = (-1)^{t_i d_i}$  into  $x_i$ .

Similarly,  $x_i^{d_i}$  contributes  $\log x_i \log y_i$  into  $\widetilde{\mathcal{W}}_K$ , and using<sup>21</sup>

$$(x; q)_n = \prod_{i=0}^{n-1} (1 - xq^i) \sim e^{\frac{1}{h}(\text{Li}_2(x) - \text{Li}_2(xq^n))}$$

we conclude that the  $q$ -Pochhammer symbols in the denominator contribute to the twisted superpotential a term:

$$(86) \quad \widetilde{\mathcal{W}}_K = \sum_i (\text{Li}_2(y_i) - \text{Li}_2(1)) + \dots$$

For example, using these (by now standard) rules, for the right-handed trefoil we get

$$(87) \quad \begin{aligned} \widetilde{\mathcal{W}}_{3_1^-} = & \log y_1 \log y_2 + \log y_2 \log y_3 + \frac{1}{2}(\log y_3)^2 + \frac{1}{2}(\log y_4)^2 + \\ & + \log y_1 \log x + \log y_2 \log ax + (\log y_3 + \log y_4) \log(-ax) + \sum_{i=1}^4 (\text{Li}_2(y_i) - \text{Li}_2(1)). \end{aligned}$$

The field content and interactions of the corresponding 3d  $\mathcal{N} = 2$  theory can be conveniently summarized in a quiver diagram illustrated in Figure 12, where, as usual, a circle denotes a gauge node and a square represents a global (flavour) symmetry.

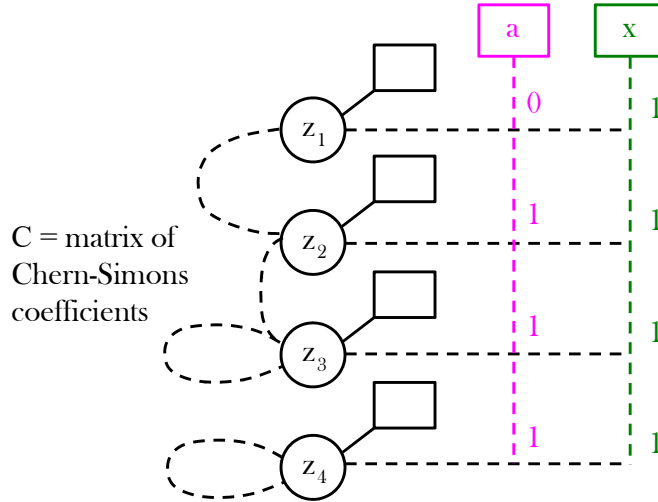


FIGURE 12. The field content and interactions of 3d  $\mathcal{N} = 2$  theory corresponding to the trefoil knot can be conveniently encoded in a quiver form. Solid lines represent charged matter fields, whereas dashed lines represent Chern-Simons couplings.

The space of vacua in this theory has three “branches” (in terminology of the  $a = 1$  specialization):

$$(88) \quad \text{spurious} : x - 1, \quad \text{abelian} : y - 1, \quad \text{non-abelian} : yx^3 + 1,$$

<sup>21</sup> $\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ , so that  $\frac{d}{d \log x} \text{Li}_2(x) = -\log(1 - x)$  and  $\text{Li}_2(1) = \frac{\pi^2}{6}$ .

where the first, “spurious”, branch comes with multiplicity 2. When  $a \neq 1$ , one of the spurious branches recombines with the abelian and non-abelian branches into a single irreducible component. The other spurious branch becomes  $ax - 1 = 0$ .

From the quiver of 3d  $\mathcal{N} = 2$  theory or from the corresponding twisted superpotential we can easily read off the anomaly coefficients, i.e. the effective Chern-Simons couplings. The symmetry  $U(1)_a$  is a symmetry of 5d sector, and to account for the anomalies of this symmetry, we need to incorporate into considerations the “closed string” 5d sector that so far has been ignored. On the other hand, the symmetry  $U(1)_x$  is a global symmetry of the “open string” sector associated with the Lagrangian brane, i.e. with a 3d surface operator in 5d bulk theory. All fields charged under this symmetry are part of the 3d  $\mathcal{N} = 2$  theory  $T[M_K]$ , and therefore we focus on the anomaly of  $U(1)_x$  symmetry first. Since the symmetry  $U(1)_x$  is preserved by the 2d  $(0, 2)$  boundary conditions that are used either in cutting and gluing or in computation of  $\widehat{Z}$ -invariants, we expect that the effective Chern-Simons coupling for the  $U(1)_x$  should vanish:

$$(89) \quad k_{xx} = 0.$$

Below we check that this anomaly vanishes on all branches.

In general, suppose that for a knot  $K$  the corresponding  $F_K$  can be written in the quiver form (83). This means that 3d  $\mathcal{N} = 2$  theory has  $U(1)^m$  gauge symmetry with one chiral multiplet per each  $U(1)$  factor and lots of Chern-Simons couplings for the dynamical gauge fields, as well as for the global symmetries  $U(1)_a$  and  $U(1)_x$ . We are especially interested in the latter, which can be computed as follows (see [GGP16] for details). First, from the data of the matrix  $C$  we construct the “dual” matrix  $G$  by writing a quadratic form<sup>22</sup>

$$(90) \quad \sum_{i,j=1}^m C_{ij} u_i u_j - \frac{1}{2} \sum_{i=1}^m u_i^2 + 2 \sum_{i=1}^m u_i \kappa_i$$

and completing the squares for variables  $u_i$ . In other words, extremizing this quadratic form with respect to  $u_i$  gives  $m$  linear equations, from which we determine  $u_i$  and substitute the solution back into (90). This gives a quadratic (Gaussian) expression for the global symmetries,

$$(91) \quad \sum_{ij} G_{ij} \kappa_i \kappa_j,$$

where  $\kappa_i$  should be understood as the fugacity  $\log x_i$ . In other words,

$$(92) \quad \begin{aligned} \kappa_i &= n_i \log x + a_i \log a + \pi i t_i \\ &= n_i \kappa + a_i \alpha + \pi i t_i, \end{aligned}$$

where there is no summation over label  $i$  and for convenience we introduced  $(\kappa, \alpha)$  in place of  $(\log x, \log a)$ . Substituting (92) into (91), we get the desired quadratic polynomial in  $\kappa$  and  $\alpha$ :

$$k_{xx} \kappa^2 + k_{ax} \alpha \kappa + k_{aa} \alpha^2 + \dots,$$

whose coefficients are effective Chern-Simons terms for  $U(1)_a$  and  $U(1)_x$  global symmetries. In particular, according to (89), we expect that  $k_{xx} = 0$ .

<sup>22</sup>This anomaly polynomial summarizes Chern-Simons couplings, including the effective Chern-Simons coupling  $-\frac{1}{2}$  for each dynamical  $U(1)$  gauge factor that comes from integrating out a charged chiral multiplet [GGP16].

For example, for the abelian and non-abelian branches of the trefoil we get<sup>23</sup>

$$\mathbf{3}_1 : \quad G_{\text{ab}} = \begin{pmatrix} 10 & 4 & -8 & 0 \\ 4 & 2 & -4 & 0 \\ -8 & -4 & 6 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad G_{\text{nab}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -6 & 4 & 8 \\ 0 & 4 & -2 & -4 \\ 0 & 8 & -4 & -10 \end{pmatrix},$$

which, together with the data of the vectors  $\mathbf{a}$ ,  $\mathbf{t}$  and  $\mathbf{n}$ , lead to the anomalies

$$0 \cdot \kappa^2 - 12\alpha\kappa - 2\alpha^2 - 16\pi i\kappa - 4\pi^2$$

for the abelian branch, and

$$0 \cdot \kappa^2 + 8\alpha\kappa + 0 \cdot \alpha^2 - 16\pi i\kappa + 12\pi i\alpha + 20\pi^2$$

for the non-abelian branch. In both cases (89) holds, as expected.<sup>24</sup>

Note, the term  $k_{xx}$  on which we focused here is a very concrete combination of the data  $C$ ,  $\mathbf{a}$ ,  $\mathbf{t}$  and  $\mathbf{n}$ :

$$k_{xx} = \sum_{ij} G_{ij} n_i n_j.$$

Moreover, from the physical point of view, extremization with respect to  $u_i$  variables in (90) comes from integrating over dynamical  $U(1)^m$  gauge degrees of freedom. Therefore, the dual matrix  $G$  is basically the inverse matrix to  $C - \frac{1}{2}\mathbf{1}$ , so the anomaly vanishing condition (89) can be stated as

$$\mathbf{n} \cdot \frac{1}{C - \frac{1}{2}\mathbf{1}} \cdot \mathbf{n} = 0.$$

Note, this anomaly vanishing condition imposes a constraint on  $\mathbf{n}$  when only  $C$  is known *a priori*. It says that  $\mathbf{n}$  is a null vector of the inverse matrix to  $C - \frac{1}{2}\mathbf{1}$  and should hold for any branch of the theory and any choice of  $C$ . Similarly, expressing  $\mathcal{W}_K$  in terms of  $C$ ,  $\mathbf{a}$ ,  $\mathbf{t}$ ,  $\mathbf{n}$  as above, and identifying  $y = \exp \frac{d\widetilde{\mathcal{W}}_K}{d \log x}$  on different branches gives a set of “invariants,” i.e. combinations of  $C$ ,  $\mathbf{a}$ ,  $\mathbf{t}$ ,  $\mathbf{n}$  that have the same value on different branches of the  $A$ -polynomial curve.

**7.2. Runaway vacua.** In (88) we glossed over one important phenomenon: while the assignment of branches makes sense only for  $a = 1$  specialization, actually it is not true that  $a = 1$  specialization of the superpotential (87) has all three branches as its critical points. In the limit  $a \rightarrow 1$  the entire branch runs off to infinity.

Indeed, the critical points of (87) are solutions to the following algebraic equations:

$$(93) \quad \begin{aligned} xy_2 &= 1 - y_1, & ay_3(y_1 - 1) &= 1 - y_3, \\ axy_1y_3 &= 1 - y_2, & -axy_4 &= 1 - y_4. \end{aligned}$$

The first equation is linear in  $y_2$ ; it has a unique and simple solution for  $y_2$  in terms of  $y_1$ . Similarly, the third equation is linear in  $y_3$  and also gives  $y_3$  as a function of  $y_1$ . The fourth

<sup>23</sup>Note the symmetry between these two matrices. This is probably an accident and does not hold for general knots; in particular, we have many more branches for general knots. On the other hand, anomaly vanishing (or matching) discussed here are expected to hold for all knots.

<sup>24</sup>Note, the coefficients of terms linear in  $\kappa$  also match on the two branches. This, however, has no significance since these terms describe mod 2 anomaly, which automatically vanishes in both cases. In other words, we do have a matching of this mod 2 discrete anomaly on both branches, but it is less impressive. Also, note that all anomalies listed here vanish in the  $\mathfrak{sl}_2$  specialization, i.e. when  $a = 1$  or  $\alpha = 0$ .

equation also has a unique solution. Substituting the resulting expressions for  $y_2$ ,  $y_3$  and  $y_4$  into the second equation we get a quadratic equation for  $y_1$ , when  $a \neq 1$  is generic.

However, when  $a = 1$ , it becomes a linear equation. It is easy to see that the second solution of this equation runs off to infinity (in  $\mathbb{C}^*$  where all our variables are valued) when  $a \rightarrow 1$ :

$$(94) \quad y_1 \simeq \frac{1-x}{x^2-x+1}(a-1) + O((a-1)^2).$$

At the same time,  $y_3$  also goes off to infinity, so that the product  $y_1 y_3$  remains finite

$$(95) \quad y_1 y_3 \simeq \frac{x-1}{x^2} + \dots$$

As a result, the whole non-abelian branch is completely missing in the theory (87) when  $a = 1$ . In order to keep this branch as part of the vacua, we need to keep  $a$  close to unity, but  $a \neq 1$ .

Note, in the superpotential (87) some of the terms that involve  $y_1$  and  $y_3$  can be manifestly grouped into terms that depend only on the product  $y_1 y_3$  and, therefore, are non-singular in the limit  $a \rightarrow 1$ . The remaining terms that can be potentially singular are the following:

$$(96) \quad \frac{1}{2}(\log y_3)^2 + \log y_3 \log(-a) + \text{Li}_2(y_1) + \text{Li}_2(y_3).$$

Writing  $z = y_3$ ,  $y_1 = \frac{A}{z}$  and using the dilogarithm reflection property

$$\text{Li}_2\left(\frac{1}{z}\right) = -\text{Li}_2(z) - \frac{\pi^2}{6} - \frac{1}{2}\log^2(-z),$$

we can write the potentially singular terms as

$$\begin{aligned} \frac{1}{2}\log^2(-z) + \log z \log a - \frac{1}{2}\log^2(-1) + \text{Li}_2\left(\frac{A}{z}\right) + \text{Li}_2(z) &= \\ &= -\frac{\pi^2}{6} + \frac{\pi^2}{2} + \log z \log a + \text{Li}_2\left(\frac{A}{z}\right) - \text{Li}_2\left(\frac{1}{z}\right). \end{aligned}$$

Since  $z \sim (1-a)^{-1}$ , the term  $\log z \log a$  goes to zero in the limit  $a \rightarrow 1$ . Therefore, we only need to estimate  $\text{Li}_2\left(\frac{A}{z}\right) - \text{Li}_2\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$  or, equivalently,  $\text{Li}_2(Aw) - \text{Li}_2(w)$  as  $w \rightarrow 0$ . Differentiating with respect to  $w$ , we see that

$$\lim_{w \rightarrow 0} \frac{1}{w} \log \frac{1-w}{1-Aw} = A - 1.$$

Therefore, it follows that  $\lim_{z \rightarrow \infty} \left[ \text{Li}_2\left(\frac{A}{z}\right) - \text{Li}_2\left(\frac{1}{z}\right) \right] \rightarrow \text{const}$  and it is then easy to check (say, numerically) that the constant is zero. We conclude that all potentially singular terms in the superpotential (96) cancel each other up to a finite contribution  $\frac{\pi^2}{3}$ .

**7.3. Closed 3-manifolds.** For a surgery on a knot  $K$ , we have the relation

$$(97) \quad \widetilde{\mathcal{W}}(S_{-p/r}^3(K)) = \frac{r}{p}(\log y)^2 + \log y \log x + \widetilde{\mathcal{W}}_K(x),$$

where  $y$  is a new variable that came from  $q^d$  in the surgery formula for  $\widehat{Z}$ -invariants. This is consistent with the surgery formula for the twisted superpotential in [GGP16, GMP16].

We wish to apply a surgery formula to a general expression of the form

$$(98) \quad F_K(x, a, q) = \sum_{n \geq 0} f_n(q) \cdot x^n = \sum_{\mathbf{d}} q^{\frac{1}{2} \mathbf{d} C \mathbf{d} + \mathbf{d} \cdot \mathbf{B} + A} \prod_i \frac{x_i^{d_i}}{(q)_{d_i}}$$

with  $x_i = (-1)^{t_i} q^{l_i} a^{a_i} x^{n_i}$ . Specializing to  $a = q^2$  and then replacing  $x \rightarrow q^{-1}x$ , we get

$$F_K(q^{-1}x, q) = \sum_{n \geq 0} f_n(q) \cdot (q^{-1}x)^n = \sum_{\mathbf{d}} q^{\frac{1}{2} \mathbf{d} C \mathbf{d} + A} \prod_i \frac{(-1)^{t_i d_i} q^{(2a_i + l_i + B_i - n_i) d_i} x^{n_i d_i}}{(q)_{d_i}},$$

where, of course,  $n = \sum_i n_i d_i$ .

Now we are ready to plug this expression in the above surgery formula with  $n = \sum_i n_i d_i$ :

$$(99) \quad \widehat{Z}(S^3_{-1/r}(K)) = \sum_{\mathbf{d}} q^{\frac{1}{2} \mathbf{d} C \mathbf{d} + A + r n^2 + r n - 2n + \frac{r}{4} + \frac{1}{4r} - \frac{3}{2}} \left[ q - q^{2r+2nr} - q^{2+2n} + q^{3+2n+2r+2nr} \right] \\ \times \prod_i \frac{(-1)^{t_i d_i} q^{(2a_i + l_i + B_i) d_i}}{(q)_{d_i}}.$$

For example, for the right-handed trefoil we have

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} A &= 0, \\ \mathbf{B} &= (0, 0, 0, 0), \\ \mathbf{t} &= (0, 0, 1, 1), \\ \mathbf{l} &= (1, 0, -\frac{1}{2}, -\frac{1}{2}), \\ \mathbf{a} &= (0, 1, 1, 1), \\ \mathbf{n} &= (1, 1, 1, 1), \end{aligned}$$

and the above surgery formula gives

$$\begin{aligned} \widehat{Z}(S^3_{-1}(\mathbf{3}_1^r)) &= 1 - q - q^5 + q^{10} - q^{11} + q^{18} + q^{30} - q^{41} + q^{43} - q^{56} + \dots, \\ \widehat{Z}(S^3_{-1/2}(\mathbf{3}_1^r)) &= q^{1/8} (1 - q - q^{11} + q^{16} - q^{23} + q^{30} + q^{60} - q^{71} + q^{85} + \dots), \\ \widehat{Z}(S^3_{-1/3}(\mathbf{3}_1^r)) &= q^{1/3} (1 - q - q^{17} + q^{22} - q^{35} + q^{42} + q^{90} - q^{101} + q^{127} + \dots), \\ &\vdots \end{aligned}$$

that agree with the earlier calculations up to overall powers of  $q$ .

**7.4. Nahm sums and MTC[ $M_3$ ].** Using the familiar rules (85)–(86), we can quickly read off the twisted superpotential from the explicit expression (99). For the trefoil we get

$$(100) \quad \widetilde{\mathcal{W}}(S^3_{-1/r}(\mathbf{3}_1^r)) = \log y_1 \log y_2 + \log y_2 \log y_3 + \frac{1}{2}(\log y_3)^2 + \frac{1}{2}(\log y_4)^2 \\ + r \left( \sum \log y_i \right)^2 + (\log y_3 + \log y_4) \log(-1) + \sum_{i=1}^4 (\text{Li}_2(y_i) - \text{Li}_2(1))$$

Note, the quiver of the corresponding 3d  $\mathcal{N} = 2$  theory  $T[S^3_{-1/r}(\mathbf{3}_1^r)]$  is very similar to the one shown in Figure 12. Namely, it is a  $U(1)^4$  gauge theory with lots of Chern-Simons couplings and one charged chiral per each  $U(1)$  gauge factor.

Also note that (100) here is in agreement with the general surgery formula (97). Indeed,  $\log y$  appears only quadratically in (97). Extremizing with respect to  $\log y$ , we generate

a quadratic term for  $\log x$ , which, in turn, appears in  $\widetilde{\mathcal{W}}_{\mathbf{3}_1}$  only linearly,<sup>25</sup> multiplying  $\sum_i \log y_i$ . Then integrating out  $\log x$  gives  $r(\sum \log y_i)^2 = r(\hbar m)^2$ .

Now let us consider a topological Landau-Ginzburg (LG) model [Vaf91, MP06] with the (twisted) superpotential (100). Its chiral ring and all partition / correlation functions are controlled by the critical points,

$$y_i^{(\lambda)} : \quad \exp \left( \frac{\partial \widetilde{\mathcal{W}}}{\partial \log y_i} \right) = 1 \quad \forall i.$$

These are the celebrated Bethe ansatz equations (BAEs) that in the context of gauge/Bethe and 3d-3d correspondence take the familiar exponentiated form (as opposed to the ‘‘2d version’’  $\frac{\partial \widetilde{\mathcal{W}}}{\partial y_i} = 0$ ). For a general expression of the form (99) these BAEs take the form

$$(101) \quad 1 - y_i = (-1)^{t_i} \prod_j y_j^{C_{ij}^{(2r)}} \quad (\forall i \text{ fixed}),$$

where we used  $\frac{d}{d \log x} \text{Li}_2(x) = -\log(1-x)$  and  $C^{(2r)}$  is the matrix introduced earlier that combines all terms in  $\widetilde{\mathcal{W}}(S^3_{-1/r}(K))$  quadratic in  $\log y_i$ . Note that these equations generalize the so-called Nahm equations [Nah07]:

$$1 - y_i = \prod_j y_j^{C_{ij}} \quad (\forall i \text{ fixed}),$$

and reduce to them when all  $t_i = 0$ . Already in the simple example of the trefoil knot, we have  $\mathbf{t} = (0, 0, 1, 1)$ , so we have to consider a more general version (101). In the trefoil case, we get the explicit form of the BAEs using (84):

$$\begin{aligned} 1 - y_3^{-1} &= 1 - y_1, & y_1 y_3 (1 - y_4^{-1}) &= 1 - y_2, \\ y_2 (1 - y_4^{-1}) &= 1 - y_1, & y_1^{2r} y_2^{2r+1} y_3^{2r} y_4^{2r} &= 1 - y_1. \end{aligned}$$

These equations, however, have no solutions  $y_i \in \mathbb{C}^*$ . We have already seen this phenomenon in (93). One way to go around it is to work with  $a \approx 1$  and then take the limit  $a \rightarrow 1$  at the end of the calculation.

The relevant equations are

$$y = y_1 y_2 y_3 y_4, \quad x = y^{2r}, \quad (1 - y)(1 + x^3 y) = 0,$$

and

$$\begin{aligned} x y_2 &= 1 - y_1, & -a x y_2 y_3 &= 1 - y_3, \\ a x y_1 y_3 &= 1 - y_2, & -a x y_4 &= 1 - y_4. \end{aligned}$$

From the previous discussion we know that  $y_2$ ,  $y_3$ , and  $y_4$  are uniquely determined in terms of  $y_1$  which, in turn, obeys a quadratic equation. The non-abelian branch of the  $A$ -polynomial

<sup>25</sup>Specializing  $a \rightarrow q^2 \rightarrow 1$  in (87) gives

$$\log y_1 \log x + \log y_2 \log a x + (\log y_3 + \log y_4) \log(-a x) \rightarrow \log x \left( \sum \log y_i \right) + (\log y_3 + \log y_4) \log(-1)$$

corresponds to a choice of solution (94) that escapes “to infinity” in the limit  $a \rightarrow 1$ :

$$\begin{aligned} y_1 &\simeq \frac{1-x}{1-x+x^2}(a-1) + \dots & y_3 &\simeq \frac{(x-1-x^2)}{x^2(a-1)}, \\ y_2 &\simeq \frac{1}{x}, & y_4 &= \frac{1}{1-ax}. \end{aligned}$$

Substituting these into (100), we find

$$\begin{aligned} \widetilde{\mathcal{W}}(S_{-1/r}^3(\mathbf{3}_1^r)) &= \text{Li}_2\left(\frac{1}{1-x}\right) + \text{Li}_2\left(\frac{1}{x}\right) + r \log^2\left(-\frac{1}{x^3}\right) + \log\left(\frac{x-1}{x^2}\right) \log\left(\frac{1}{x}\right) \\ &\quad + \frac{1}{2} \log\left(\frac{1}{1-x}\right) \left(\log\left(\frac{1}{1-x}\right) + 2\pi i\right) - \frac{\pi^2}{3}, \end{aligned}$$

where we took advantage of (95) and the analysis below it. This twisted superpotential can be brought to the form

$$\begin{aligned} \widetilde{\mathcal{W}}(S_{-1/r}^3(\mathbf{3}_1^r)) &= r \log\left(\frac{-1}{x^3}\right)^2 + \log(x)^2 + \frac{1}{2} \log(-x)^2 - \pi i \log(-x) - \frac{\pi^2}{3} \\ &= r \log(y)^2 + \log(x)^2 + \frac{1}{2} \log(-x)^2 - \pi i \log(-x) - \frac{\pi^2}{3} \end{aligned}$$

from which classical Chern-Simons values can be seen at critical points of  $\widetilde{\mathcal{W}}$ .

This identification is a part of much richer structure. Namely, it was proposed in [GPV17] that to each 3d  $\mathcal{N} = 2$  theory one can associate a braided tensor category of line operators, which in many cases is a modular tensor category.<sup>26</sup> For 3d  $\mathcal{N} = 2$  theories  $T[M_3]$ , it is denoted  $\text{MTC}[M_3]$ . The Grothendieck ring of this category is the Jacobi ring of the Landau-Ginzburg model with the superpotential  $\widetilde{\mathcal{W}}$ . In particular, critical points of  $\widetilde{\mathcal{W}}$  correspond to simple objects of  $\text{MTC}[M_3]$ , and various data – such as matrix elements of  $S$  and  $T$  matrices, conformal dimensions, effective central charge  $c_{\text{eff}}$ , etc. – are determined by this effective Landau-Ginzburg model:

$$(102) \quad T_{\lambda\mu} = \delta_{\lambda\mu} e^{\widetilde{\mathcal{W}}(\lambda)}, \quad S_{0\lambda} = e^{-U} \det \text{Hess } \widetilde{\mathcal{W}} \Big|_{\lambda}.$$

As a result, the twisted partition function on a genus- $g$  surface  $\Sigma_g$  can be written in the standard form [Vaf91, MP06]:

$$Z(S^1 \times \Sigma_g) = \sum_{\lambda} (S_{0\lambda})^{2-2g} = \sum_{\text{crit. pts. of } \widetilde{\mathcal{W}}} \left( e^{-U} \det \text{Hess } \widetilde{\mathcal{W}} \right)^{g-1},$$

with  $S_{0\lambda}$  given by (102).

The effective central charge  $c_{\text{eff}}$  can be read off from the asymptotic growth of the coefficients  $a_n$  in the expansion of  $\widehat{Z}$ -invariants, which are identified with the characters of logarithmic vertex operator algebras (log-VOAs):

$$\chi_b(q) = \widehat{Z}_b(q) = q^{\Delta_b} \sum_n a_n q^n,$$

<sup>26</sup>See [FG20, DGN+20, CGK20, CCF+19] for further discussion and applications.

where it is important that we include in  $\widehat{Z}_a(q)$  the  $(q)_\infty$  denominator associated with the center of mass chiral multiplet of the  $T[M_3]$  theory [CCF<sup>+</sup>19]. Specifically, as  $n \rightarrow \infty$ ,

$$(103) \quad a_n \sim \exp 2\pi \sqrt{\frac{1}{6}c_{\text{eff}} n}$$

For example, for small surgeries on the right-handed trefoil knot we have

$$\widehat{Z}_0(S_{-1/r}^3(\mathbf{3}_1)) = \frac{1}{\eta(q)} \left( \widetilde{\Psi}_{36r+6}^{(6r-5)} - \widetilde{\Psi}_{36r+6}^{(6r+7)} - \widetilde{\Psi}_{36r+6}^{(30r-1)} + \widetilde{\Psi}_{36r+6}^{(30r+11)} \right)$$

and, therefore, from (102) and (103) we read off

$$c_{\text{eff}} = c - 24h_{\text{min}} = 1 \quad \text{and} \quad c = 1 - \frac{(36r + 5)^2}{6r + 1}.$$

Since in this case we actually know the precise log-VOA whose characters can be identified with  $\widehat{Z}$ -invariants, we can check that these indeed match with the VOA central charges for all values of  $r \in \mathbb{Z}_+$ .

The situation is a little more interesting in cases where the precise log-VOA has not been identified yet. In those cases, we can use (102) and (103) to obtain its modular data, central charges and conformal dimensions  $h_\lambda$  related to  $T_{\lambda\lambda} = e^{2\pi i(h_\lambda - \frac{c}{24})}$ . The simplest class of such examples is a family of small surgeries on the figure-eight knot  $S_{-1/r}^3(\mathbf{4}_1)$ . For example, for  $r = 2$  we have

$$(104) \quad \widehat{Z}(S_{-1/2}^3(\mathbf{4}_1)) = \frac{q^{-\frac{1}{2}}}{\eta(q)} (1 - q + 2q^3 - 2q^6 + q^9 + 3q^{10} + q^{11} - q^{14} - 3q^{15} + \dots \\ \dots - 15040q^{500} + \dots).$$

Note, without the Dedekind  $\eta$ -function in the denominator, this would be a  $q$ -series with oscillation coefficients of a growing amplitude. However, with  $\eta(q)$  in the denominator, all coefficients  $a_n$  have the same sign, as expected for a character of a (logarithmic) VOA. Moreover, their rate of growth (103), illustrated in Figure 13, is consistent with

$$c_{\text{eff}} \approx 1$$

which suggests that  $c_{\text{eff}} = 1$ , as in the case of small surgeries on the trefoil.

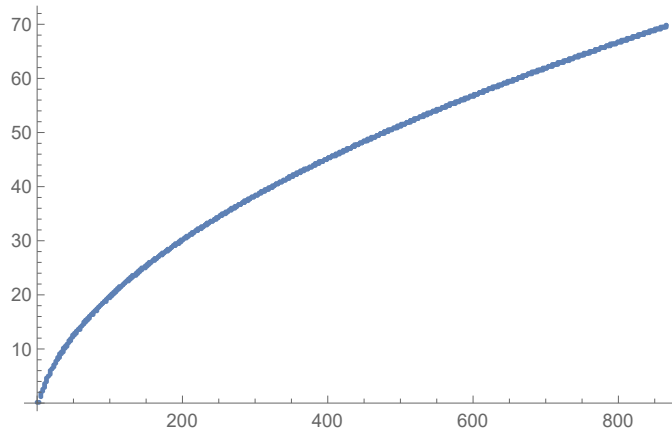


FIGURE 13. Plot of  $\log a_n$  as a function of  $n$  for  $-\frac{1}{2}$  surgery on the figure-eight knot.

## 8. FUTURE DIRECTIONS

In this section, we provide a summary of interesting open problems that emerged during our research:

- Is there a way to unify the wave functions  $F_K^{(\alpha)}(x, a, q)$  associated to different branches? Recall that the wave functions  $F_K^{(\alpha)}(x, a, q)$  are associated to branches on the boundary  $(x, a) = (0, 0)$  (or via Weyl symmetry,  $(x, a) = (\infty, \infty)$ ). In order to unify them, we probably need to understand better what happens in the middle of the holomorphic Lagrangian.

An interesting observation in this direction is that for  $\mathfrak{sl}_2$  the non-abelian branch  $F_K$  invariants seem to appear as a certain limit of 3d index from [DGG14].

**Conjecture 10.** *Let  $K$  be a knot and let  $\alpha$  denote a non-degenerate, non-abelian branch of the  $A$ -polynomial of slope  $-\frac{1}{p}$ . That is,  $F_K^\alpha$  has a prefactor of the form  $e^{\frac{p(\log x)^2}{2h}}$ . Let's write (unreduced)  $F_K^\alpha(x, q)$  as*

$$F_K^\alpha(x, q) = e^{\frac{\frac{p}{2}(\log x)^2 + \epsilon \log(-1) \log x}{h}} x^d \sum_{j=0}^{\infty} f_j^\alpha(q) x^j$$

for some  $\epsilon \in \{0, 1\}$  and  $d \in \frac{1}{2}\mathbb{Z}$ . Then  $f_j^\alpha(q)$  can be obtained from  $\mathcal{I}_K(m, e)$  by

$$f_j^\alpha(q) = \lim_{m \rightarrow \infty} (-1)^{(\epsilon+1)m} q^{-d(j+1)m} \mathcal{I}_K(m, pm + j).$$

For instance, for the figure-eight knot, with  $p = 2$ , we empirically checked that

$$\begin{aligned} \lim_{m \rightarrow \infty} (-1)^m q^{-\frac{1}{2}m} \mathcal{I}_{\mathcal{A}_1}(m, 2m) &= 1, \\ \lim_{m \rightarrow \infty} (-1)^m q^{-\frac{2}{2}m} \mathcal{I}_{\mathcal{A}_1}(m, 2m + 1) &= -q - q^2 - q^3 - q^4 - q^5 - q^6 - \dots \\ &= \frac{1 - 2q}{1 - q} - 1, \\ \lim_{m \rightarrow \infty} (-1)^m q^{-\frac{3}{2}m} \mathcal{I}_{\mathcal{A}_1}(m, 2m + 2) &= -q - q^2 + q^5 + q^6 + 2q^7 + \dots \\ &= \frac{1 - 3q - q^2 + 4q^3}{(1 - q)(1 - q^2)} - \frac{1 - 2q}{1 - q}, \end{aligned}$$

so they match up perfectly. If true, this conjecture can be seen as a way to unify non-abelian branch  $F_K$  invariants in  $\mathfrak{sl}_2$  case.

In recent works of [GGM21a, GGM21b], authors studied resurgence in complex Chern-Simons theory and identified the entries of the Stokes matrix between non-abelian flat connections with the 3d index from [DGG14]. Moreover, the abelian branch  $F_K$  is related to the first row of the Stokes matrix. This is a strong encouragement to study resurgence in the  $a$ -deformed setting, in particular looking for a generalization of 3d index that unifies  $F_K$  from all the branches.

- How are quivers associated to different branches related to each other? So far we only understand how they are related in a rather indirect way; e.g. they give rise to the same holomorphic Lagrangian after eliminating variables from the quiver  $A$ -polynomials. It would be nice to understand how to get from a quiver from one branch to a quiver from another branch. Probably, understanding of the previous

question (i.e. unification of  $F_K$ 's for various branches) will lead to an answer to this question and vice versa.

- It would be interesting to understand the categorification of  $F_K$  invariants in the context of quivers. Recent results for knot conormals [EKL21] suggest that the presence of quiver nodes with  $x_i \sim y^{-n}$ ,  $n > 1$  imply a different structure of the refined generating function. Another subtlety lies in the fact that the  $t$ -deformation inherited from the superpolynomials and super- $A$ -polynomials is not consistent with the one guided by the number of loops. Further complication comes from the fact that different forms of knot complement quivers would lead to different  $t$ -deformations.
- It would be desirable to find explicit formulas for  $F_K$  invariants in quiver forms for knots and branches not covered in Section 4.2 and Appendix A. The case of non-abelian branch of slope  $\infty$  for knot  $5_2$ , partly discussed in Appendix A, seems to be especially interesting.

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## APPENDIX A. QUIVERS FOR COMPLEMENTS OF VARIOUS KNOTS

In this appendix we present formulas for  $F_K$  invariants for all knots with 5 or 6 crossings, as well as (3,4) torus knot, obtained using the method presented in Section 4.1. For simplicity, we will keep  $q$ -Pochhammers  $(x; q^{-1})_{(\dots)}$  in the concise form. In case the quiver form of  $F_K$  is needed, they can be easily expanded in two ways, as discussed in Section 4.1.

**5<sub>1</sub> knot.** The  $F_K$  invariant for the abelian branch of 5<sub>1</sub> is given by

$$\begin{aligned} F_{\mathfrak{5}_1}(x, a, q) &= \sum_{\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4} (-1)^{\tilde{d}_2 + \tilde{d}_4} a^{\tilde{d}_2 + \tilde{d}_4} q^{\frac{1}{2}(2\tilde{d}_1 - \tilde{d}_2 + 4\tilde{d}_3 + \tilde{d}_4)} \\ &\quad \times x^{\tilde{d}_1 + \tilde{d}_2 + 3\tilde{d}_3 + 3\tilde{d}_4} q^{\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} \times \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_4}}{\prod_{i=1}^4 (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & -2 \\ -1 & 0 & -2 & -1 \end{pmatrix}.$$

Expanding the  $q$ -Pochhammer  $(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_4}$  one can check that this formula is consistent with Equation (54).

The quiver form of the  $F_K$  for the mirror image  $m(\mathfrak{5}_1) = \mathfrak{5}_1^r$  and its non-abelian branch with slope  $-\frac{1}{5}$  (corresponding to framing  $f = 5$ ) is given by

$$\begin{aligned} F_{\mathfrak{5}_1^r}^{(-\frac{1}{5})}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_4} (-1)^{\tilde{d}_3} a^{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_4} \\ &\quad \times q^{\frac{1}{2}(\tilde{d}_1 - \tilde{d}_2 + 2\tilde{d}_3 - 3\tilde{d}_4)} x^{\tilde{d}_1 + \tilde{d}_2} q^{-\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} \\ &\quad \times q^{(3\tilde{d}_1 + 3\tilde{d}_2 + 4\tilde{d}_3 + 4\tilde{d}_4) \sum_i \tilde{d}_i} q^{-\frac{5}{2}(\sum_i \tilde{d}_i)^2} q^{-\sum_{i < j} \tilde{d}_i \tilde{d}_j} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_4}}{\prod_{i=1}^4 (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix}.$$

**5<sub>2</sub> knot.** The  $F_K$  invariant for 5<sub>2</sub> corresponding to the abelian branch reads:

$$\begin{aligned} F_{\mathfrak{5}_2}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_6} (-1)^{\tilde{d}_2 + \tilde{d}_3 + \tilde{d}_6} a^{\tilde{d}_2 + \tilde{d}_4 + \tilde{d}_5 + 2\tilde{d}_6} q^{\frac{1}{2}(2\tilde{d}_1 - \tilde{d}_2 + \tilde{d}_3 - 2\tilde{d}_5 - 3\tilde{d}_6)} \\ &\quad \times x^{\tilde{d}_1 + \tilde{d}_2 + 2\tilde{d}_4 + \tilde{d}_6} q^{\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_6}}{\prod_{i=1}^6 (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 & 3 \end{pmatrix}.$$

For the mirror image  $m(\mathfrak{5}_2) = \mathfrak{5}_2^r$  we need to consider it in framing  $f = 5$  and we get  $F_K$  for the non-abelian branch corresponding to slope  $-\frac{1}{5}$ :

$$\begin{aligned} F_{\mathfrak{5}_2^r}^{(-\frac{1}{5})}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_6} (-1)^{\tilde{d}_3 + \tilde{d}_4} a^{2\tilde{d}_1 + 2\tilde{d}_2 + \tilde{d}_3 + 2\tilde{d}_4 + \tilde{d}_5 + \tilde{d}_6} \\ &\times q^{\frac{1}{2}(-5\tilde{d}_1 - 3\tilde{d}_2 - 2\tilde{d}_3 - 4\tilde{d}_4 - 3\tilde{d}_5 - \tilde{d}_6)} x^{2\tilde{d}_1 + 3\tilde{d}_2 + \tilde{d}_3 + 2\tilde{d}_4 + \tilde{d}_6} q^{-\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} \\ &\times q^{(2\tilde{d}_1 + \tilde{d}_2 + 3\tilde{d}_3 + 2\tilde{d}_4 + 4\tilde{d}_5 + 3\tilde{d}_6) \sum_i \tilde{d}_i} q^{-\frac{5}{2} (\sum_i \tilde{d}_i)^2} q^{-\sum_{i < j} \tilde{d}_i \tilde{d}_j} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_6}}{\prod_{i=1}^6 (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 1 & 3 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 2 & 4 & 3 \\ 1 & 0 & 2 & 1 & 3 & 2 \end{pmatrix}.$$

For knots for which  $f_0 \neq 1$ , some of the coefficients of  $\tilde{d}_i$  in the exponent of  $x$  are equal to 0. For example, let us take  $\mathfrak{5}_2$  for infinite slope branch, and try to find quivers. We have

$$f_0^{\text{sl}_N} = \frac{1}{(q)_{N-2}} \sum_{i=0}^{N-2} \begin{bmatrix} N-2 \\ i \end{bmatrix}.$$

As we know, we can rewrite this in quiver form:

$$f_0^{\text{sl}_N} = \sum_{i=0}^{N-2} \frac{1}{(q)_i (q)_{N-2-i}} = \sum_{i=0}^{\infty} \frac{(q^{N-1-i}; q)_{\infty}}{(q)_i (q)_{\infty}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^j q^{Nj} q^{-j-ij} q^{\frac{1}{2}(j^2-j)} q^k}{(q)_i (q)_j (q)_k},$$

and so

$$f_0(a, q) = \sum_{\tilde{d}_1, \tilde{d}_2, \tilde{d}_3} \frac{(-1)^{\tilde{d}_2} a^{\tilde{d}_2} q^{-\tilde{d}_2 - \tilde{d}_1 \tilde{d}_2} q^{\frac{1}{2}(\tilde{d}_2^2 - \tilde{d}_2)} q^{\tilde{d}_3}}{(q)_{\tilde{d}_1} (q)_{\tilde{d}_2} (q)_{\tilde{d}_3}}.$$

This fixes the first  $3 \times 3$  block of the quiver matrix and the first 3 entries of the rows of  $\mathbf{a}, \mathbf{l}$ . For further ones, let us denote

$$a_N := f_0^{\text{sl}_N} \left( = \sum_{i,j,k} \frac{(-1)^j q^{Nj} q^{-j-ij} q^{\frac{1}{2}(j^2-j)} q^k}{(q)_i (q)_j (q)_k} \right), \quad b_N := f_1^{\text{sl}_N}.$$

Then we have the recursion

$$(1 - q^N) a_{N+2} = 2a_{N+1} - a_N, \quad (1 - q) b_N = (1 - q^{N-1})(a_N + a_{N+1}).$$

This suggests that we should have (at least) 4 nodes whose  $\tilde{d}_i$ 's have coefficient 1 in the exponent of  $x$  in  $F_K$ . The following formula (that is relevant for the off-diagonal entries in the quiver for the expressions in  $f_1(a, q)$  and higher ones) helps in simplifying the expressions:

**Lemma 1.** *For integers  $\alpha, \beta, \gamma \geq 0$  we have:*

$$\sum_{i,j,k} q^{\alpha i + \beta j + \gamma k} \frac{(-1)^j q^{Nj} q^{-j-ij} q^{\frac{1}{2}(j^2-j)} q^k}{(q)_i (q)_j (q)_k} = (q)_\gamma \sum_{i=0}^{N-2+\beta} \frac{q^{\alpha i}}{(q)_i (q)_{N-2+\beta-i}}.$$

In addition, if we denote

$$P_{\alpha,\beta}(N) = \sum_{i=0}^{N-2+\beta} \frac{q^{\alpha i}}{(q)_i (q)_{N-2+\beta-i}},$$

then it can be obtained recursively by

$$P_{0,\beta}(N) = a_{N+\beta},$$

and

$$P_{\alpha,\beta}(N) = a_{N+\beta} - \sum_{\delta=0}^{\alpha-1} q^\delta P_{\delta,\beta-1}, \quad \alpha > 0.$$

In particular:

$$(105) \quad \sum_{i,j,k} q^{\beta j + \gamma k} \frac{(-1)^j q^{Nj} q^{-j-ij} q^{\frac{1}{2}(j^2-j)} q^k}{(q)_i (q)_j (q)_k} = (q)_\gamma a_{N-2+\beta}.$$

In consequence, the first guess for the first  $7 \times 7$  block can be:

$$C = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} \mathbf{a} &= (0, 1, 0, 0, 0, 1, 1), \\ \mathbf{l} &= (0, -1, 1, 0, 0, -1, -1), \\ \mathbf{n} &= (0, 0, 0, 1, 1, 1, 1). \end{aligned}$$

By using Lemma 1 (especially (105)), one can check that we get the correct answer for  $f_1^{sl_N}$ . However, one can see that the  $f_2^{sl_N}$  is not quite correct, but it is close. Therefore either we should either change  $C$  (choose different off-diagonal entries), or add more nodes/generators. They can also be added with different flavours: some of them proportional to  $x$ , and maybe some of them proportional to  $x^2$ .

**6<sub>1</sub> knot.** For the knot  $\mathbf{6}_1$ , the  $F_K$  invariant corresponding to the non-abelian branch with slope  $-\frac{1}{2}$  is given by

$$\begin{aligned} F_{\mathbf{6}_1}^{(-\frac{1}{2})}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_8} (-1)^{\tilde{d}_3 + \tilde{d}_4 + \tilde{d}_6 + \tilde{d}_7} a^{\tilde{d}_1 + 2\tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4 + 3\tilde{d}_5 + 2\tilde{d}_6 + 2\tilde{d}_7 + \tilde{d}_8} \\ &\times q^{\frac{1}{2}(-2\tilde{d}_1 - 4\tilde{d}_2 - 3\tilde{d}_3 - \tilde{d}_4 - 6\tilde{d}_5 - 5\tilde{d}_6 - 3\tilde{d}_7 - 2\tilde{d}_8)} x^{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_4 + 2\tilde{d}_5 + \tilde{d}_6 + 2\tilde{d}_7} \\ &\times q^{\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j + (\tilde{d}_1 + \tilde{d}_2 + 2\tilde{d}_3 + \tilde{d}_4 + \tilde{d}_6 + 2\tilde{d}_8) \sum_i \tilde{d}_i - (\sum_i \tilde{d}_i)^2} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_8}}{\prod_{i=1}^8 (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 1 & 2 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 & -1 \\ 0 & 2 & 1 & 2 & 4 & 2 & 3 & 1 \\ -1 & 0 & -1 & 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 3 & 2 & 3 & 1 \\ -1 & -1 & -2 & -1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

For the mirror image  $m(\mathbf{6}_1) = \mathbf{6}_1^r$  and for non-abelian branch of slope  $-\frac{1}{4}$  corresponding to framing  $f = 4$ , the  $F_K$  invariant is given by

$$\begin{aligned} F_{\mathbf{6}_1^r}^{(-\frac{1}{4})}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_8} (-1)^{\tilde{d}_3 + \tilde{d}_4 + \tilde{d}_6 + \tilde{d}_7} a^{2\tilde{d}_1 + \tilde{d}_2 + 2\tilde{d}_3 + 2\tilde{d}_4 + 3\tilde{d}_5 + \tilde{d}_6 + \tilde{d}_7 + 2\tilde{d}_8} \\ &\times q^{\frac{1}{2}(-4\tilde{d}_1 - 2\tilde{d}_2 - 3\tilde{d}_3 - 5\tilde{d}_4 - 6\tilde{d}_5 - \tilde{d}_6 - 3\tilde{d}_7 - 4\tilde{d}_8)} x^{3\tilde{d}_1 + \tilde{d}_2 + 2\tilde{d}_3 + \tilde{d}_4 + 3\tilde{d}_5 + \tilde{d}_6 + 2\tilde{d}_8} \\ &\times q^{-\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} \\ &\times q^{(2\tilde{d}_2 + \tilde{d}_3 + 2\tilde{d}_4 + 2\tilde{d}_6 + 3\tilde{d}_7 + \tilde{d}_8) \sum_i \tilde{d}_i} q^{-2(\sum_i \tilde{d}_i)^2} q^{-\sum_{i < j} \tilde{d}_i \tilde{d}_j} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_8}}{\prod_{i=1}^8 (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 2 & 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 0 & -2 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -2 & -1 & -2 & -1 & 0 & -2 \\ -1 & 0 & -1 & 1 & -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 3 & 1 \\ -1 & -1 & -2 & -1 & -2 & 0 & 1 & 0 \end{pmatrix}.$$

**6<sub>2</sub> knot.** For the knot  $\mathbf{6}_2$  in framing  $f = 2$ , the  $F_K$  invariant corresponding to a non-abelian branch with slope  $-\frac{1}{2}$ , is given by

$$\begin{aligned} F_{\mathbf{6}_2}^{(-\frac{1}{2})}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_{10}} (-1)^{\tilde{d}_1 + \tilde{d}_4 + \tilde{d}_5 + \tilde{d}_8 + \tilde{d}_9} a^{\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_4 + \tilde{d}_5 + \tilde{d}_6 + \tilde{d}_9 + 2\tilde{d}_7 + 2\tilde{d}_8 + 2\tilde{d}_{10}} \\ &\times q^{\frac{1}{2}(-3\tilde{d}_1 - 2\tilde{d}_2 + 2\tilde{d}_3 - \tilde{d}_4 - \tilde{d}_5 - 4\tilde{d}_7 - 3\tilde{d}_8 + \tilde{d}_9 - 2\tilde{d}_{10})} x^{\tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4 + \tilde{d}_5 + 2\tilde{d}_6 + \tilde{d}_7 + 3\tilde{d}_8 + 4\tilde{d}_9 + 3\tilde{d}_{10}} \\ &\times q^{\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j + (2\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4 + \tilde{d}_5 + \tilde{d}_7 - \tilde{d}_8 - \tilde{d}_9 - \tilde{d}_{10}) \sum_i \tilde{d}_i - (\sum_i \tilde{d}_i)^2} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_{10}}}{\prod_{i=1}^{10} (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \\ 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \end{pmatrix}.$$

For the mirror image  $m(\mathbf{6}_2) = \mathbf{6}_2^r$  in framing  $f = 4$ , we get the following  $F_K$  invariant corresponding to non-abelian branch of slope  $-\frac{1}{4}$ :

$$\begin{aligned} F_{\mathbf{6}_2^r}^{(-\frac{1}{4})}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_{10}} (-1)^{\tilde{d}_2 + \tilde{d}_5 + \tilde{d}_6 + \tilde{d}_9 + \tilde{d}_{10}} a^{2\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + 2\tilde{d}_4 + \tilde{d}_5 + \tilde{d}_6 + \tilde{d}_7 + \tilde{d}_{10}} \\ &\times q^{\frac{1}{2}(-2\tilde{d}_1 + \tilde{d}_2 - 4\tilde{d}_4 - \tilde{d}_5 - \tilde{d}_6 - 2\tilde{d}_7 + 2\tilde{d}_8 + \tilde{d}_9 - 3\tilde{d}_{10})} x^{2\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + 2\tilde{d}_4 + \tilde{d}_5 + \tilde{d}_6 + \tilde{d}_7} \\ &\times q^{-\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} q^{\tilde{d}_1 + 2\tilde{d}_2 + 2\tilde{d}_3 + \tilde{d}_4 + 2\tilde{d}_5 + 2\tilde{d}_6 + 2\tilde{d}_7 + 3\tilde{d}_8 + 3\tilde{d}_9 + 3\tilde{d}_{10}} \sum_i \tilde{d}_i q^{-2(\sum_i \tilde{d}_i)^2} \\ &\times q^{-\sum_{i < j} \tilde{d}_i \tilde{d}_j} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_{10}}}{\prod_{i=1}^{10} (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} -2 & -2 & -1 & -1 & -1 & -1 & 0 & -1 & 1 & 1 \\ -2 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 1 & 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \end{pmatrix}.$$

**$\mathbf{6}_3$  knot.** We note that knot  $\mathbf{6}_3$  is amphichiral. For  $\mathbf{6}_3$  in framing  $f = 3$ , we obtain the following  $F_K$  invariant corresponding to a non-abelian branch for slope  $-\frac{1}{3}$ :

$$\begin{aligned} F_{\mathbf{6}_3}^{(-\frac{1}{3})}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_{12}} (-1)^{\tilde{d}_2 + \tilde{d}_6 + \tilde{d}_7 + \tilde{d}_9 + \tilde{d}_{12}} a^{\tilde{d}_1 + 2\tilde{d}_2 + \tilde{d}_3 + \tilde{d}_4 + 2\tilde{d}_5 + \tilde{d}_6 + \tilde{d}_7 + 2\tilde{d}_9 + \tilde{d}_{10} + \tilde{d}_{11}} \\ &\times q^{\frac{1}{2}(-\tilde{d}_1 - 4\tilde{d}_2 - \tilde{d}_3 - 3\tilde{d}_4 - 3\tilde{d}_5 - 2\tilde{d}_7 + \tilde{d}_8 - 2\tilde{d}_9 + \tilde{d}_{10} - \tilde{d}_{11} + 2\tilde{d}_{12})} \\ &\times x^{2\tilde{d}_1 + \tilde{d}_2 + \tilde{d}_3 + 2\tilde{d}_5 + 2\tilde{d}_6 + \tilde{d}_7 + 2\tilde{d}_9 + 2\tilde{d}_{10} + \tilde{d}_{11} + \tilde{d}_{12}} \\ &\times q^{\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} + (\tilde{d}_1 + 2\tilde{d}_2 + 2\tilde{d}_3 + 3\tilde{d}_4 + \tilde{d}_5 + \tilde{d}_6 + 2\tilde{d}_7 + 3\tilde{d}_8 + \tilde{d}_9 + \tilde{d}_{10} + 2\tilde{d}_{11} + 2\tilde{d}_{12}) \sum_i \tilde{d}_i - \frac{3}{2} (\sum_i \tilde{d}_i)^2 \\ &\times \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_{12}}}{\prod_{i=1}^{12} (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & -2 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -2 & 0 & -1 & -2 & -3 & -1 & 0 & -2 & -2 \\ 0 & 1 & 1 & 0 & 2 & 1 & 0 & -1 & 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 2 & 1 & 1 & 0 \\ -1 & -1 & 0 & -2 & 0 & 0 & -1 & -2 & 0 & 0 & -1 & -2 \\ -1 & -2 & -2 & -3 & -1 & -1 & -2 & -2 & 0 & -1 & -1 & -2 \\ 0 & 1 & 1 & -1 & 2 & 2 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & -1 & 2 & 2 & 1 & 0 \\ -1 & 0 & 0 & -2 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & -2 & -1 & 0 & -2 & -2 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

**8<sub>19</sub> knot.** For **8<sub>19</sub>**, which is (3, 4)-torus knot, the corresponding  $F_K$  invariant for abelian branch is given by

$$\begin{aligned} F_{\mathbf{8}_{19}}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_{10}} (-1)^{\tilde{d}_5 + \tilde{d}_6 + \tilde{d}_7 + \tilde{d}_8 + \tilde{d}_9} a^{\tilde{d}_5 + \tilde{d}_6 + \tilde{d}_7 + \tilde{d}_8 + \tilde{d}_9 + 2\tilde{d}_{10}} \\ &\times q^{\frac{1}{2}(2\tilde{d}_1 + 2\tilde{d}_2 + 4\tilde{d}_3 + 6\tilde{d}_4 - \tilde{d}_5 - \tilde{d}_6 + \tilde{d}_7 + \tilde{d}_8 + 3\tilde{d}_9 - 2\tilde{d}_{10})} x^{\tilde{d}_1 + 2\tilde{d}_2 + 3\tilde{d}_3 + 5\tilde{d}_4 + \tilde{d}_5 + 2\tilde{d}_6 + 3\tilde{d}_7 + 4\tilde{d}_8 + 5\tilde{d}_9 + 4\tilde{d}_{10}} \\ &\times q^{\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} q^{(-\tilde{d}_1 - 2\tilde{d}_2 - 3\tilde{d}_3 - 5\tilde{d}_4 - \tilde{d}_5 - 2\tilde{d}_6 - 3\tilde{d}_7 - 4\tilde{d}_8 - 5\tilde{d}_9 - 4\tilde{d}_{10}) \sum_i \tilde{d}_i} \\ &\times \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_{10}}}{\prod_{i=1}^{10} (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 2 & 3 & 3 & 5 & 2 & 3 & 3 & 5 & 5 & 5 \\ 3 & 4 & 4 & 5 & 3 & 4 & 4 & 5 & 5 & 5 \\ 3 & 4 & 4 & 5 & 4 & 4 & 4 & 6 & 5 & 6 \\ 5 & 5 & 5 & 6 & 6 & 5 & 6 & 6 & 6 & 6 \\ 2 & 3 & 4 & 6 & 3 & 3 & 4 & 5 & 6 & 5 \\ 3 & 4 & 4 & 5 & 3 & 5 & 4 & 6 & 5 & 6 \\ 3 & 4 & 4 & 6 & 4 & 4 & 5 & 6 & 6 & 6 \\ 5 & 5 & 6 & 6 & 5 & 6 & 6 & 7 & 7 & 7 \\ 5 & 5 & 5 & 6 & 6 & 5 & 6 & 7 & 7 & 7 \\ 5 & 5 & 6 & 6 & 5 & 6 & 6 & 7 & 7 & 8 \end{pmatrix}.$$

For the mirror image  $m(\mathbf{8}_{19}) = \mathbf{8}_{19}^r$ , we obtain  $F_K$  invariant corresponding to the non-abelian branch of slope  $-\frac{1}{6}$ :

$$\begin{aligned} F_{\mathbf{8}_{19}^r}^{f=8}(x, a, q) &= \sum_{\tilde{d}_1, \dots, \tilde{d}_{10}} (-1)^{\tilde{d}_6 + \tilde{d}_7 + \tilde{d}_8 + \tilde{d}_9 + \tilde{d}_{10}} a^{2\tilde{d}_1 + 2\tilde{d}_2 + 2\tilde{d}_3 + 2\tilde{d}_4 + 2\tilde{d}_5 + \tilde{d}_6 + \tilde{d}_7 + \tilde{d}_8 + \tilde{d}_9 + \tilde{d}_{10}} \\ &\times q^{\frac{1}{2}(-2\tilde{d}_1 - 4\tilde{d}_2 - 4\tilde{d}_3 - 6\tilde{d}_4 - 8\tilde{d}_5 - \tilde{d}_6 - \tilde{d}_7 - 3\tilde{d}_8 - 3\tilde{d}_9 - 5\tilde{d}_{10})} x^{3\tilde{d}_1 + 2\tilde{d}_2 + 2\tilde{d}_3 + \tilde{d}_4 + \tilde{d}_5 + 2\tilde{d}_6 + \tilde{d}_7 + \tilde{d}_8} \\ &\times q^{-\frac{1}{2} \sum_{i,j} \tilde{C}_{ij} \tilde{d}_i \tilde{d}_j} q^{(-2\tilde{d}_1 - \tilde{d}_2 - \tilde{d}_3 - \tilde{d}_6 + \tilde{d}_9 + \tilde{d}_{10}) \sum_i \tilde{d}_i} \\ &\times q^{-\sum_{i < j} \tilde{d}_i \tilde{d}_j} \frac{(x; q^{-1})_{\tilde{d}_1 + \dots + \tilde{d}_{10}}}{\prod_{i=1}^{10} (q)_{\tilde{d}_i}}, \end{aligned}$$

where

$$\tilde{C} = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 3 & 5 & 2 & 3 & 3 & 5 & 5 \\ 2 & 3 & 4 & 4 & 5 & 3 & 4 & 4 & 5 & 5 \\ 3 & 3 & 4 & 4 & 5 & 4 & 4 & 4 & 6 & 5 \\ 5 & 5 & 5 & 5 & 6 & 6 & 5 & 6 & 6 & 6 \\ 1 & 2 & 3 & 4 & 6 & 3 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 4 & 5 & 3 & 5 & 4 & 6 & 5 \\ 3 & 3 & 4 & 4 & 6 & 4 & 4 & 5 & 6 & 6 \\ 4 & 5 & 5 & 6 & 6 & 5 & 6 & 6 & 7 & 7 \\ 5 & 5 & 5 & 5 & 6 & 6 & 5 & 6 & 7 & 7 \end{pmatrix}.$$

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