3-manifolds and q-series

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Overview of the talk

This is a gentle introduction to conjectural 3-manifold invariants, $\hat{Z}$ and F_K, valued in q-series with integer coefficients

$$\begin{aligned} Y^3 \quad &\rightsquigarrow \quad \hat{Z}_b(Y;q), \text{ a } q\text{-series} \\ S^3 \setminus K \quad &\rightsquigarrow \quad F_K(x,q), \text{ a two-variable series} \end{aligned}$$

conjectured by Gukov-Putrov-Vafa, Gukov-Pei-Putrov-Vafa, and Gukov-Manolescu.

I will review what is known about these conjectural invariants, and then discuss some recent developments.

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   - Alexander polynomial
   - Colored Jones polynomials
   - WRT invariant
2. q-series for 3-manifolds (GPPV and GM)
   - Motivation behind these conjectural invariants
   - GPPV series $\hat{Z}$
   - GM series F_K
3. Some generalizations
   - Generalization to arbitrary gauge group
   - HOMFLY-PT analogs

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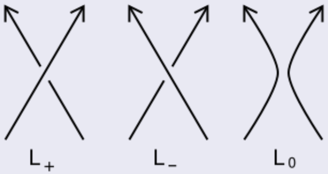
Definition

The *Alexander polynomial* of a link L, $\Delta_L(x) \in \mathbb{Z}[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]$, is defined by the following skein relations :

$$\Delta_O = 1$$

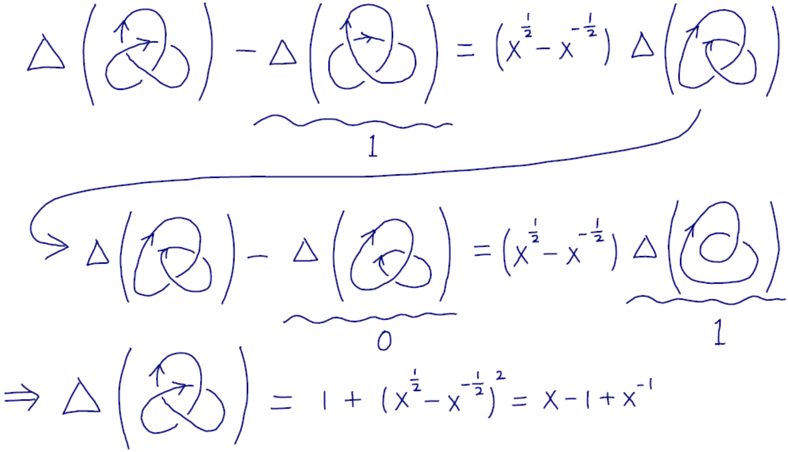
$$\Delta_{L_+} - \Delta_{L_-} = (x^{\frac{1}{2}} - x^{-\frac{1}{2}})\Delta_{L_0}$$

where



Background : Alexander polynomial, continued

For example,



Background : Alexander polynomial, continued

Just a few remarks

- For knots, $\Delta_K(x) \in \mathbb{Z}[x, x^{-1}]$ and $\Delta_K(x) = \Delta_K(x^{-1})$.
- $\Delta_L(x) = 0$ for split links.
- From representation theoretic point of view, Alexander polynomial can be defined from the Burau representation of the braid group.

Background : colored Jones polynomials

Definition

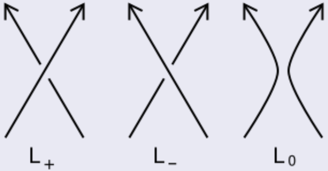
The *(unreduced) Jones polynomial* of a link L, $\tilde{J}_2(L;q) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, is defined by the following skein relations :

$$\tilde{J}_2(L_1 \sqcup L_2) = \tilde{J}_2(L_1)\tilde{J}_2(L_2)$$

$$\tilde{J}_2(O) = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$$

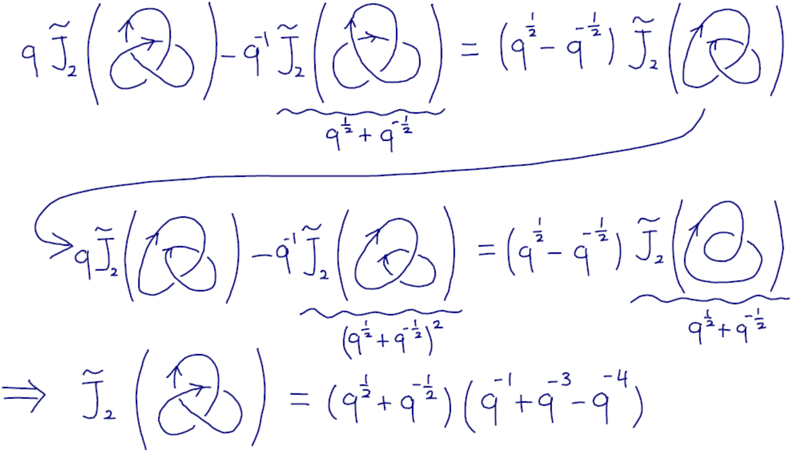
$$q\tilde{J}_2(L_+) - q^{-1}\tilde{J}_2(L_-) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{J}_2(L_0)$$

where



Background : colored Jones polynomials, continued

For example,



Let $\Delta_n(d)$ be Chebyshev polynomials defined recursively by

$$\Delta_1 = 1, \quad \Delta_2(d) = d, \quad \Delta_{n+1}(d) = d\Delta_n(d) - \Delta_{n-1}(d).$$

Let's write $\Delta_n(d) = \sum_{0\le j\le n-1} c_{n,j} d^j$.

Definition

The *(unreduced) n-colored Jones polynomial* of a link L can be defined by

$$\tilde{J}_n(L;q) = \sum_{0\le j\le n-1} c_{n,j} \tilde{J}_2(L^j;q) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$$

where L^j is the j-th cabling of L.

Background : colored Jones polynomials, continued

Remarks

- Really, these invariants are coming from the representation theory of quantum groups. One should think that the strands of L are colored by V_n, the irreducible n-dimensional representation of $\mathfrak{sl}_2$ (or more precisely, $U_q(\mathfrak{sl}_2)$).
- $\tilde{J}_n(O; q) = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} := [n]$.
- So far, we assumed that K is 0-framed. $\tilde{J}_n$ for the p-framed knot has an extra factor of $q^{p\frac{n^2-1}{4}}$.

Definition

The *reduced n-colored Jones polynomial* of a knot K is

$$J_n(K; q) := \frac{\tilde{J}_n(K; q)}{[n]} \in \mathbb{Z}[q, q^{-1}]$$

Recall that

- Every 3-manifold is a Dehn surgery on a framed link in S^3 (Lickorish-Wallace theorem).
- Two Dehn surgeries on framed links give the same 3-manifold iff they are related via a sequence of Kirby moves (or Fenn-Rourke moves).

It turns out that when q is a root of unity, certain linear combinations of the colored Jones polynomials of a framed link are invariant under Kirby moves, and hence give 3-manifold invariants.

Background : WRT invariant, continued

Let Y be a 3-manifold obtained as a surgery on a framed link $L \subset S^3$.

Definition

Fix a level $k \in \mathbb{Z}_{\geq 3}$, and let $q = e^{\frac{2\pi i}{k}}$. Set $\omega = \sum_{1 \leq n \leq k-1} S_{1n} V_n$, a linear combination of colors. Then the *Witten-Reshetikhin-Turaev (WRT) invariant* for Y is defined as

$$WRT(Y; e^{\frac{2\pi i}{k}}) = S_{11} C^{\sigma(L)} \tilde{J}_\omega(L)$$

Here

$$S_{mn} = \sqrt{\frac{2}{k}} \sin\left(\frac{mn\,\pi}{k}\right),$$

and $C = \exp\left(-\pi i \frac{3(k-2)}{4k}\right)$ is a framing factor.

Also define $\tau(Y; e^{\frac{2\pi i}{k}}) := \frac{WRT(Y; e^{\frac{2\pi i}{k}})}{WRT(S^3; e^{\frac{2\pi i}{k}})}$.

Background : WRT invariant, continued

Remarks

- This is a special case of the construction of 3-manifold invariants from modular tensor categories due to Reshetikhin and Turaev.
- Although we have used primitive roots of unity in the definition, WRT invariants can be defined at every root of unity.
- From physics point of view, colored Jones polynomials and WRT invariants are partition functions in Chern-Simons theory.
- So far we have assumed that the gauge group of the Chern-Simons theory is $SU(2)$, but everything can be generalized to any simply-connected semisimple Lie group, such as $SU(N)$.

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Motivation 1 : analytic continuation of WRT

Let $P = \Sigma(2,3,5) = S^3_{-1}(\mathbf{3}^l_1)$ be the Poincare homology sphere.

Theorem (Lawrence-Zagier '99)

For every root of unity ξ,

$$\tau(P;\xi) = \lim_{q\to\xi} \frac{\hat{Z}_0(P;q)}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}$$

where

$$\begin{aligned}\hat{Z}_0(P;q) &= q^{-\frac{3}{2}}(2 - \sum_{n\geq 0} q^n (q^n)_n) \\ &= q^{-\frac{3}{2}}(1 - q - q^3 - q^7 + q^8 + q^{14} + \cdots)\end{aligned}$$

They also showed similar results for three-fibered Seifert integer homology spheres.

Motivation 2 : categorification

Khovanov categorified the (colored) Jones polynomials

$$J_n(K;q) = \sum_{i,j} (-1)^i q^j \dim \mathcal{H}_n^{i,j}(K)$$

Question

What are the "categorifiable objects" for 3-manifolds, analogous to colored Jones polynomials for links?

One possible candidate : conjectural q-series invariants I'm about to discuss.

Gukov-Pei-Putrov-Vafa (GPPV) series $\hat{Z}$ for 3-manifolds

A few years ago, Gukov-Putrov-Vafa [GPV] and Gukov-Pei-Putrov-Vafa [GPPV] conjectured the existence of a new invariant "$\hat{Z}$" for 3-manifolds.

They are analytic continuation of WRT invariants in a certain sense, and conjecturally they admit a categorification.

$$\hat{Z}_b(Y;q) = \sum_{i,j} (-1)^i q^j \dim \mathcal{H}^{i,j}_{BPS}(Y;b)$$

GPPV series $\hat{Z}$ for 3-manifolds, continued

For simplicity, we assume that Y is a rational homology 3-sphere.

Conjecture (GPPV'17, improved in GM'19)

There exist functions

$$\hat{Z} : b \mapsto \hat{Z}_b(q) \in q^{\Delta_b}\mathbb{Z}[[q]], \quad \Delta_b \in \mathbb{Q}$$

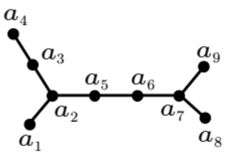
where b ranges over $\mathrm{Spin}^c(Y)/\mathbb{Z}_2$. These q-series $\hat{Z}_b(q)$ decomposes the WRT invariant in the following sense :

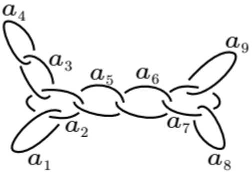
$$WRT(Y, e^{\frac{2\pi i}{k}}) = \sum_{a,b} e^{2\pi i k \cdot \ell k(a,a)} X_{ab} \hat{Z}_b(q)\Big|_{q \to e^{\frac{2\pi i}{k}}}$$

where $X_{ab} = \frac{e^{2\pi i \ell k(a,b)} + e^{-2\pi i \ell k(a,b)}}{|\mathcal{W}_a||\mathcal{W}_b|\sqrt{|H_1(Y)|}}$ *and* $a \in H_1(Y;\mathbb{Z})/\mathbb{Z}_2$.

$\exists$ a definition of $\hat{Z}$ for weakly negative definite plumbings [GPPV, GM].

Given a tree Γ with vertices decorated by integers, there is a natural link associated to it, and the corresponding *plumbed 3-manifold* $Y(\Gamma)$ is the one obtained by the surgery on that link.





GPPV series $\hat{Z}$ for 3-manifolds, continued

Definition (GPPV'17, improved in GM'19)

For $Y(\Gamma)$ with a weakly negative-definite linking matrix M,

$$\hat{Z}_b(Y;q) = (-1)^\pi q^{\frac{3\sigma - \mathrm{Tr}\, M}{4}} \; v.p. \int_{|x_v|=1} \prod_{v\in V} \frac{dx_v}{2\pi i x_v} (x_v^{\frac{1}{2}} - x_v^{-\frac{1}{2}})^{2-\deg v} \Theta_b^{-M}(x,q)$$

where

$$b \in (2\mathbb{Z}^V + \delta)/2M\mathbb{Z}^V \cong \mathrm{Spin}^c(Y),$$

$$\delta_v = 2 - \deg v,$$

and

$$\Theta_b^{-M}(x,q) := \sum_{\ell \in 2M\mathbb{Z}^V + b} q^{-\frac{(\ell, M^{-1}\ell)}{4}} \prod_{v\in V} x_v^{\ell_v}$$

Note that $\hat{Z}_b = \hat{Z}_{-b}$, so really $b \in \mathrm{Spin}^c(Y)/\mathbb{Z}_2$.

GPPV series $\hat{Z}$ for 3-manifolds, continued

It's not complicated! What it really means is

1. Start from the integrand $\prod_{v\in V}(x_v^{\frac{1}{2}} - x_v^{-\frac{1}{2}})^{2-\deg v}$
2. Expand it "symmetrically", e.g.

$$(x^{\frac{1}{2}} - x^{-\frac{1}{2}})^{-1} = \frac{1}{2}\left(\cdots + x^{-\frac{3}{2}} + x^{-\frac{1}{2}} - x^{\frac{1}{2}} - x^{\frac{3}{2}} - \cdots\right)$$

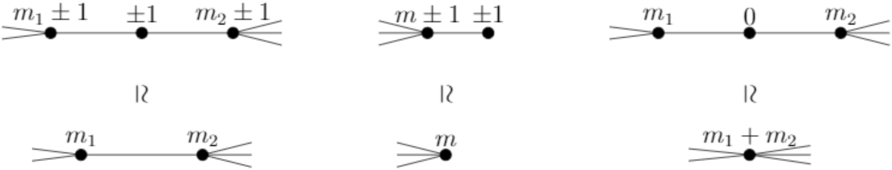
3. Apply "Laplace transform"

$$\prod_{v\in V} x_v^{\ell_v} \mapsto \begin{cases} q^{-\frac{(\ell, M^{-1}\ell)}{4}} & \text{if } \ell \in 2M\mathbb{Z}^V + b \\ 0 & \text{otherwise} \end{cases},$$

and up to normalization we get $\hat{Z}_b(Y; q)$.

GPPV series $\hat{Z}$ for 3-manifolds, continued

It is known that $Y(\Gamma) = Y(\Gamma')$ iff Γ and Γ' are related via a sequence of *Neumann moves*.



Theorem (See GM'19)

$\hat{Z}_b$ is invariant under Neumann moves.

GPPV series $\hat{Z}$ for 3-manifolds, continued

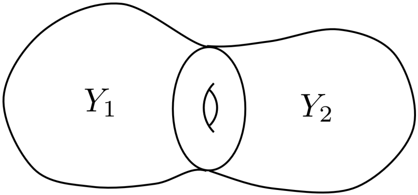
Examples :

$$\hat{Z}(\Sigma(2,3,5)) = q^{-3/2}(1 - q - q^3 - q^7 + q^8 + q^{14} + q^{20} + q^{29} - \cdots)$$
$$= q^{-3/2}\left(2 - \sum_{n\geq 0} q^n (q^n)_n\right)$$

$$\hat{Z}(\Sigma(2,3,7)) = q^{1/2}(1 - q - q^5 + q^{10} - q^{11} + q^{18} + q^{30} - q^{41} + \cdots)$$
$$= q^{1/2}\sum_{n\geq 0} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^{n+1})_n}$$

GM series F_K for knot complements

In 2019, Gukov and Manolescu [GM] studied an analog of $\hat{Z}$ for knot complements (and 3-manifolds with toral boundaries), as well as their behavior under Dehn surgery (and gluing along toral boundaries).



Gukov-Manolescu (GM) series F_K for knot complements, continued

The following theorem was conjectured by Melvin and Morton and was proved by Bar-Natan and Garoufalidis and by Rozansky.

Theorem (Bar-Natan-Garoufalidis '96, Rozansky '96)

The colored Jones polynomials have the following asymptotic expansion

$$J_n(K; q = e^{\hbar}) = \sum_{j \geq 0} \frac{P_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}$$
$$= \frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\hbar + \frac{P_2(x)}{\Delta_K(x)^5}\frac{\hbar^2}{2!} + \cdots$$

where $P_j(x) \in \mathbb{Z}[x, x^{-1}]$, $P_0 = 1$, *and* $x = q^n = e^{n\hbar}$.

GM series F_K for knot complements, continued

For simplicity, let K be a proper knot in S^3. Then their conjecture can be summarized as follows :

Conjecture (Gukov-Manolescu'19)

There exists a two-variable series

$$F_K(x,q) = \frac{1}{2}\sum_{\substack{m\geq 1\\ m \text{ odd}}} f_m(q)(x^{\frac{m}{2}} - x^{-\frac{m}{2}}), \quad f_m(q) \in \mathbb{Z}[q^{-1}, q]]$$

such that its perturbative series (asymptotic expansion) agrees with that of Melvin-Morton expansion

$$F_K(x, e^{\hbar}) = (x^{\frac{1}{2}} - x^{-\frac{1}{2}})\left(\frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\hbar + \frac{P_2(x)}{\Delta_K(x)^5}\frac{\hbar^2}{2!} + \cdots\right).$$

GM series F_K for knot complements, continued

Moreover, there is a surgery formula

$$\hat{Z}_b(S^3_{\frac{p}{r}}(K)) \cong \mathcal{L}^{(b)}_{\frac{p}{r}}\left[(x^{\frac{1}{2r}} - x^{-\frac{1}{2r}})F_K(x,q)\right]$$

whenever the r.h.s. makes sense. Here, the Laplace transform is defined as

$$\mathcal{L}^{(b)}_{\frac{p}{r}} : x^u \mapsto \begin{cases} q^{-\frac{r}{p}u^2} & \textit{if } \frac{r}{p}u \in \mathbb{Z} + \frac{b}{p} \\ 0 & \textit{otherwise} \end{cases}$$

Additionally, the two-variable series $F_K(x,q)$ is annihilated by the quantum A-polynomial

$$\hat{A}_K(\hat{x},\hat{y},q)F_K(x,q) = 0.$$

GM series F_K for knot complements, continued

Here, the *quantum A-polynomial* is an operator

$$\hat{A}_K(\hat{x},\hat{y},q) = \sum_{0\leq j\leq d} a_j(\hat{x},q)\hat{y}^j, \quad \hat{y}\hat{x} = q\hat{x}\hat{y}$$

that annihilates the *colored Jones generating function*

$$\sum_{n\geq 0} J_n(K;q)y^{-n}$$

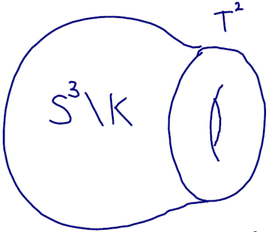
where $\hat{y}y^{-n} = y^{-n+1}$ and $\hat{x}y^{-n} = q^n y^{-n}$.

When acting on F_K, $\hat{x}x^m = x^{m+1}$ and $\hat{y}x^m = q^m x^m$.

GM series F_K for knot complements, continued

Remarks

- The quantum A-polynomial is a quantization of the classical A-polynomial describing the SL_2-character variety.



$$\mathcal{M}_{flat}(S^3\backslash K) \subset \mathcal{M}_{flat}(T^2) = \frac{\mathbb{C}^\times \times \mathbb{C}^\times}{\mathbb{Z}_2}$$

- Hence, classically, x, y are the holonomy eigenvalues along the meridian and longitude, respectively.

GM series F_K for knot complements, continued

In [GM] F_K was computed for torus knots and the figure-8 knot.

- For torus knots,

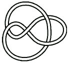
$$F_{T(s,t)}(x,q) = q^{\frac{st-s/t-t/s}{4}} \frac{1}{2} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \epsilon_m q^{\frac{m^2}{4st}} (x^{\frac{m}{2}} - x^{-\frac{m}{2}})$$

where

$$\epsilon_m = \begin{cases} 1 & \text{if } m \equiv st \pm (s-t) \mod 2st \\ -1 & \text{if } m \equiv st \pm (s+t) \mod 2st \\ 0 & \text{otherwise} \end{cases}$$

This formula was derived from a plumbing description of torus knot complements.

GM series F_K for knot complements, continued

- For the figure-8 knot $\mathbf{4}_1 =$ ,

$$F_{\mathbf{4}_1}(x,q) = \frac{1}{2} \sum_{\substack{m \geq 0 \\ m \text{ odd}}} f_m^{\mathbf{4}_1}(q)\left(x^{\frac{m}{2}} - x^{-\frac{m}{2}}\right)$$

where

$$f_1^{\mathbf{4}_1} = 1$$
$$f_3^{\mathbf{4}_1} = 2$$
$$f_5^{\mathbf{4}_1} = q^{-1} + 3 + q$$
$$f_7^{\mathbf{4}_1} = 2q^{-2} + 2q^{-1} + 5 + 2q + 2q^2$$
$$f_9^{\mathbf{4}_1} = q^{-4} + 3q^{-3} + 4q^{-2} + 5q^{-1} + 8 + 5q + 4q^2 + 3q^3 + q^4$$
$$\vdots$$

GM series F_K for knot complements, continued

A remarkable example from [GM] :

$$S^3_{-1}(\mathbf{4}_1) = S^3_{+1}(\mathbf{3}_1^l) = -S^3_{-1}(\mathbf{3}_1^r) = -S^3_{+1}(\mathbf{4}_1).$$

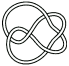
$$\begin{aligned}\hat{Z}(S^3_{-1}(\mathbf{4}_1)) &= -q^{-1/2}(1 + q + q^3 + q^4 + q^5 + 2q^7 + q^8 + 2q^9 + \cdots) \\ &= -q^{-1/2}\sum_{n\geq 0}\frac{q^{n^2}}{(q^{n+1})_n} \\ \hat{Z}(S^3_{-1}(\mathbf{3}_1^r)) &= q^{1/2}(1 - q - q^5 + q^{10} - q^{11} + q^{18} + q^{30} - q^{41} + \cdots) \\ &= q^{1/2}\sum_{n\geq 0}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q^{n+1})_n}\end{aligned}$$

The top one is Ramanujan's order 7 mock theta function, and the bottom one is the false theta function associated to it.

GM series F_K for knot complements, continued

More examples (P., work in progress) :

- Positive double twist knots $K_{n,m}$

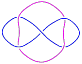
For instance, $K_{2,1} = \mathbf{5}_2 =$ 

$$F_{\mathbf{5}_2}(x,q) = \frac{1}{2} \sum_{\substack{m \geq 0 \\ m \text{ odd}}} f_m^{\mathbf{5}_2}(q)(x^{\frac{m}{2}} - x^{-\frac{m}{2}})$$

where

$$f_1^{\mathbf{5}_2} = -q^{-1} + 1 - q^2 + q^5 - q^9 + q^{14} - q^{20} + q^{27} - q^{35} + q^{44} - q^{54} + \cdots$$
$$f_3^{\mathbf{5}_2} = -q^{-1} + 1 + q - q^2 - q^3 - q^4 + q^5 + q^6 + q^7 + q^8 - q^9 - q^{10} - \cdots$$
$$\vdots$$

GM series F_K for knot complements, continued

- Whitehead link =  (our first non-trivial example of a link!)

$$F_{Wh}(x,y,q) \cong \frac{1}{4}\sum_{i,j\geq 0} f_{ij}^{Wh}(q)(x^{i+\frac{1}{2}} - x^{-i-\frac{1}{2}})(y^{j+\frac{1}{2}} - y^{-j-\frac{1}{2}})$$

where $f_{ij}^{Wh}(q) = f_{ji}^{Wh}(q)$, and $f_{ij}^{Wh}(q) =$

1	1	1	$\cdots$
1	$-q^{-1}+1+q$	$-q^{-2}-q^{-1}+1+q+q^2$	$\cdots$
1	$-q^{-2}-q^{-1}+1+q+q^2$	$-2q^{-2}-2q^{-1}+q+2q^2+q^3+q^4$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	

In the classical limit $q \to 1$, F_L converges to the inverse of the *Alexander-Conway function* $\nabla_L(x_1, \cdots, x_m)$.

GM series F_K for knot complements, continued

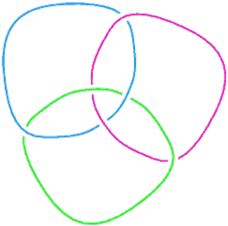
Remarks

- We can get various twist knots by performing a Dehn surgery on a component of the Whitehead link.

$$S^3_{-\frac{1}{p},\cdot}(Wh) = K_p$$

  Surgery formula works for $p \geq 1$ and $p = -1$.

- There is an algorithm to compute F_L for the Borromean rings as well.



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Generalization to arbitrary gauge group

In [P](arXiv 1909:13002) we studied a generalization of conjectures by GPPV and GM to arbitrary complex simple Lie algebras (with $\mathfrak{sl}_2$ case corresponding to the original conjectures)

Conjecture (GPPV conjecture for arbitrary gauge group)

WRT invariants with gauge group G can be decomposed into

$$WRT^G(Y, e^{\frac{2\pi i}{k}}) = \sum_{a,b} e^{2\pi ik\ \ell k(a,a)} X_{a,b} \hat{Z}^G_b(q)\Big|_{q \to e^{\frac{2\pi i}{k}}}$$

Generalization to arbitrary gauge group, continued

Conjecture (Higher rank F_K)

For any knot K and a choice of a semi-simple root system G of rank r, there exists a series

$$F_K^G(\mathbf{x}, q) = \frac{1}{|W|} \sum_{\beta \in P_+ \cap (Q+\rho)} f_\beta^G(q) \sum_{w \in W} (-1)^{l(w)} x^{w(\beta)}.$$

where $\mathbf{x} = (x_1, \cdots, x_r)$ and the coefficients $f_\beta^G(q)$ are Laurent series with integer coefficients, such that its asymptotic expansion agrees with the (higher rank) Melvin-Morton-Rozansky expansion for the (higher rank) colored Jones polynomials :

$$F_K^G(\mathbf{x}, e^\hbar) = \prod_{\alpha \in \Delta^+} (x^{\alpha/2} - x^{-\alpha/2}) \sum_{j \geq 0} \frac{P_j(\mathbf{x})}{\left(\prod_{\alpha \in \Delta^+} \Delta_K(x^\alpha)\right)^{2j+1}} \frac{\hbar^j}{j!}$$

where $P_j(\mathbf{x}) \in \mathbb{Z}[x_1, x_1^{-1}, \cdots, x_r, x_r^{-1}]$ and $P_0 = 1$.

Generalization to arbitrary gauge group, continued

Under surgery, the following should hold

$$\hat{Z}_b^G(S^3_{\frac{p}{r}}(K)) \cong \mathcal{L}^{(b)}_{\frac{p}{r}}\left[\prod_{\alpha\in\Delta^+}(x^{\frac{\alpha}{2r}} - x^{-\frac{\alpha}{2r}})F_K^G(\mathbf{x},q)\right]$$

whenever the r.h.s. makes sense.
Moreover, this series should be annihilated by the (higher rank) quantum A-polynomials :

$$\hat{A}_K(\hat{x}_1,\hat{y}_1,\cdots,\hat{x}_r,\hat{y}_r)F_K^G(\mathbf{x},q) = 0.$$

Generalization to arbitrary gauge group, continued

We have concrete results for negative definite plumbings and torus knot complements.

Definition-Theorem (P.)

For a weakly negative definite plumbing Y, we have

$$\hat{Z}^G_b(Y;q) = (-1)^{\pi|\Delta^+|} q^{\frac{3\sigma - \mathrm{Tr}\, M}{2}(\rho,\rho)} \ v.p. \int_{|x_{vi}|=1} \prod_{\substack{v \in V \\ 1 \le i \le r}} \frac{dx_{vi}}{2\pi i x_{vi}}$$

$$\left(\sum_{w \in W} (-1)^{l(w)} x_v^{w(\rho)} \right)^{2-\deg v} \Theta_b^{-M}(x,q)$$

where $b \in (Q^V + \delta)/MQ^V$ and $\delta_v = (2 - \deg v)\rho$.

This is invariant under Neumann moves.

Generalization to arbitrary gauge group, continued

From the plumbing description of torus knot complements, we get

Theorem (P.)

For torus knots $T(s,t)$, we have

$$F^G_{T(s,t)}(x,q) \cong \frac{1}{|W|} \sum_{\beta \in P_+ \cap (Q+\rho)} \sum_{w \in W} (-1)^{l(w)} x^{w(\beta)} \sum_{(w_1,w_2)\in W^2} (-1)^{l(w_1 w_2)} \mathbf{1}(\beta, w_1, w_2) N_{\frac{1}{st}(\beta + t w_1(\rho) + s w_2(\rho))} q^{\frac{(\beta,\beta)}{2st}}$$

where

$$\mathbf{1}_{(\beta,w_1,w_2)} := \begin{cases} 1 & \text{if } \frac{1}{st}(\beta + t w_1(\rho) + s w_2(\rho)) \in P_+ \cap (Q+\rho) \\ 0 & \text{otherwise} \end{cases},$$

$N_\ell = \sum_{w\in W} (-1)^{l(w)} K(w(\ell))$, *and* $K(\beta)$ *are Kostant partition functions.*

HOMFLY-PT analog $F_K(x, a, q)$

As a consequence of the generalization to arbitrary gauge group, we can consider

$$F_K^{SU(N)}(x_1, x_2, \cdots, x_{N-1}, q)$$

or more specifically its specialization to symmetric representations

$$F_K^{SU(N),sym}(x, q) := F_K^{SU(N)}(x, q \cdots, q, q)$$

HOMFLY-PT analog $F_K(x, a, q)$, continued

In a work in progress with Gruen, Gukov, Kucharski and Sulkowski, we study the HOMFLY-PT analogue of GM series F_K. We found $F_K(x, a, q)$ for torus knots $T(2, 2n+1)$ and the figure-8 knot.

Conjecture (Gruen-Gukov-Kucharski-P.-Sułkowski, in preparation)

There exists a three-variable function $F_K(x, a, q)$ such that $F_K(x, q^N, q)$ is the same as the symmetric specialization of $F_K^{SU(N)}(x, q)$. Moreover, the series $F_K(x, a, q)$ has the following properties :

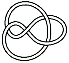
1. $\hat{A}_K(\hat{x}, \hat{y}, a, q) F_K(x, a, q) = 0$
2. $F_K(x^{-1}, a, q) = F_K(a^{-1} q^2 x, a, q)$ *(Weyl symmetry)*

3a. $F_K(x, q^N, q)\Big|_{q \to 1} = \Delta_K(x)^{1-N}$

3b. $F_K(x, q, q) = 1$

3c. $F_K(x, 1, q) = \Delta_K(q^{-1} x)$

HOMFLY-PT analog $F_K(x, a, q)$, continued

Example : figure-8 knot 

$$
\begin{aligned}
F_{\mathbf{4}_1}(x, a, q) = 1 \\
&+ \frac{-2(a-q)}{(1-q)} q^{-1} x \\
&+ \frac{(a-q)(-1-3q+(3q+q^2)a)}{(1-q)(1-q^2)} q^{-2} x^2 \\
&+ \cdots
\end{aligned}
$$

This three-variable function satisfies all the desired properties in the conjecture. In particular,

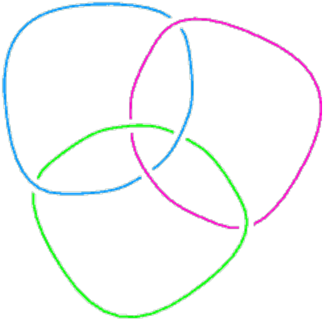
$$F_{\mathbf{4}_1}(x, q^2, q) = 1 + 2x + (q^{-1} + 3 + q)x^2 + \cdots$$

In summary ...

- Conjectural 3-manifold invariants $\hat{Z}$, F_K
- Can be generalized to any semi-simple root system
- and HOMFLY-PT version

Some exciting open questions

- Find their mathematicial definitions
- Understand their properties, e.g. how do they behave under Heegaard splitting or branched covering
- Categorify them!



Thank you for your attention!

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