# $F_K$ AND VOLUME CONJECTURE

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In this short note, we study volume conjecture-like expansions for $F_K$. Let's first fix some notations and conventions.

- Let $J_{K,n}(q)$ be the $n$-colored Jones polynomial, normalized so that $J_{K,1}(q) = 1$ and $J_{\mathrm{unknot},n}(q) = 1$ for all $n$. Define
$$J_K(y,q) = \sum_{n\geq 1} J_{K,n}(q) y^{-n}$$
to be their generating series.
- Let
$$F_K(x,q) = \sum_{n\geq 0} F_{K,n}(q) x^n$$
be the two-variable series of [GM21], normalized so that $F_{\mathrm{unknot}}(x,q) = 1$. This series is well-defined for closures of homogeneous braids [Par21], in which case the coefficients $F_{K,n}(q)$ are Laurent polynomials.

The sequence of polynomials $J_{K,n}(q)$ and $F_{K,n}(q)$ satisfy the same $q$-difference equation $\hat{A}(\hat{x},\hat{y})$, and as a result, they have a lot of similar features. However, while the colored Jones polynomials have been studied extensively, the polynomials $F_{K,n}(q)$ are relatively new and haven't been studied as much. For instance, it is well-known that one can obtain the hyperbolic volume of a knot complement from a certain asymptotics of $J_{K,n}(q)$, while the analogous statement for $F_{K,n}(q)$ hasn't appeared in the literature to the best of our knowledge. One of the motivations of this note is to fill in this gap by studying asymptotic series associated to the polynomials $F_{K,n}(q)$ analogous to that of $J_{K,n}(q)$.

From the sequence of polynomials $J_{K,n}(q)$, we can obtain various perturbative series associated to various branches of the A-polynomial curve $A_K(x,y) = 0$. This is a well-studied subject, and we summarize some of them below.

- By taking the large $n$ asymptotics of $J_{K,n}(e^{\frac{2\pi i}{n}})$, we get (in case $K$ is hyperbolic) [Guk05]
$$J_{K,n}(e^{\frac{2\pi i}{n}}) \underset{n\to\infty}{\sim} e^{\frac{V_K}{2\pi}n} n^{\frac{3}{2}} \Phi_K^{\sigma_1}\left(\frac{2\pi i}{n}\right), \tag{1}$$
where $V_K$ is the complexified volume of the knot complement, and $\Phi_K^{\sigma_1}(h)$ is the formal power series
$$\Phi_K^{\sigma_1}(h) \in \overline{\mathbb{Q}}[[h]]$$
associated to the geometric branch $y^{\sigma_1}(x)$ of the A-polynomial curve at $x = 1$. For instance,

$$\Phi_{\mathbf{4_1}}^{\sigma_1}(h) = \frac{1}{\sqrt[4]{3}}\left(1 + \frac{11}{72\sqrt{-3}}h + \frac{697}{2(72\sqrt{-3})^2}h^2 + \frac{724351}{30(72\sqrt{-3})^3}h^3 + \cdots\right) \in \frac{1}{\sqrt[4]{3}}\mathbb{Q}(\sqrt{-3})[[h]],$$

where $\mathbb{Q}(\sqrt{-3})$ is the trace field of $\mathbf{4}_1$.

When $K$ is not hyperbolic, the right-hand side of (1) is in general a trans-series, consisting of contributions of flat connections whose real part of classical action is 0. For instance, for the left-handed trefoil $\mathbf{3}_1$,

$$J_{\mathbf{3}_1,n}(e^{\frac{2\pi i}{n}}) \underset{n\to\infty}{\sim} e^{\frac{2\pi i}{24}n} n^{\frac{3}{2}} \Phi^{\sigma_1}_{\mathbf{3}_1}\left(\frac{2\pi i}{n}\right) + \Phi^{\sigma_0}_{\mathbf{3}_1}\left(\frac{2\pi i}{n}\right),$$

where

$$\Phi^{\sigma_1}_{\mathbf{3}_1}(h) = e^{-\frac{2\pi i}{8}}\left(1 - \frac{23}{2^3\cdot 3}h + \frac{529}{(2^3\cdot 3)^2}\frac{h^2}{2!} - \frac{12167}{(2^3\cdot 3)^3}\frac{h^3}{3!} + \frac{279841}{(2^3\cdot 3)^4}\frac{h^4}{4!} + \cdots\right) = e^{-\frac{2\pi i}{8}} q^{-\frac{23}{24}},$$

$$\Phi^{\sigma_0}_{\mathbf{3}_1}(h) = 1 + 0h + 2\,\frac{h^2}{2!} + 12\,\frac{h^3}{3!} + 146\,\frac{h^4}{4!} + 2580\,\frac{h^5}{5!} + 63722\,\frac{h^6}{6!} + \cdots .$$

- Melvin-Morton-Rozansky expansion gives

$$J_{K,n}(e^h) \underset{\substack{n\to\infty \\ x=e^{nh}\text{ fixed}}}{\sim} \frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\frac{h}{1!} + \frac{P_2(x)}{\Delta_K(x)^5}\frac{h^2}{2!} + \cdots , \tag{2}$$

  where $P_n(x)$ are some Laurent polynomials in $x$. This is the perturbative series associated to the trivial branch $y^{\sigma_0}(x) = 1$ of the A-polynomial curve. Since $F_K(x, q)$ is a non-perturbative completion of the Melvin-Morton-Rozansky expansion, we will often write the right-hand side of (2) as $F_K(x, e^h)$.

  Note, $\Phi^{\sigma_0}_{\mathbf{3}_1}(h)$ in the previous example is exactly $F_{\mathbf{3}_1}(1, e^h)$.

- Taking the perturbative expansion of the generating series, we get

$$J_K(y, e^h) = \frac{1}{y-1} + \frac{Q_1(y)}{(y-1)^2}\frac{h}{1!} + \frac{Q_2(y)}{(y-1)^3}\frac{h^2}{2!} + \cdots , \tag{3}$$

  where $Q_n(y)$ are some Laurent polynomials in $y$.

  There are a few ways to further specialize this series.
  - The first is to set $y = -1$:

$$J_K(-1, e^h) = -\frac{1}{2}F_K(1, e^h).$$

  - The second is to take the residue of $\frac{1}{y}J_K(y, e^h)$ either at $y = 0$ or at $y = 1$:

$$\mathrm{Res}_{y=1}\left(\frac{1}{y}J_K(y, e^h)\right) = -\mathrm{Res}_{y=0}\left(\frac{1}{y}J_K(y, e^h)\right) = F_K(1, e^h).$$

Now, we can do the same for the sequence of polynomials $F_{K,n}(q)$, completely in parallel.

- By taking the large $n$ asymptotics of $F_{K,n}(e^{\frac{2\pi i}{n}})$, in case $K$ is hyperbolic, we conjecture the following [1]

  **Conjecture 0.1** (Volume conjecture for $F_K$)**.**

$$F_{K,n}(e^{\frac{2\pi i}{n}}) \underset{n\to\infty}{\sim} e^{\frac{V_K}{2\pi}n} n^{\frac{1}{2}} \Psi^{\sigma_1}_K\left(\frac{2\pi i}{n}\right), \tag{4}$$

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[1] Note, $F_{K,n}(e^{\frac{2\pi i}{n}})$ is well-defined for *any* knot, as the coefficient of $x^n$ in the power series expansion of $\frac{\mathrm{ADO}_n(e^{-\frac{2\pi i}{n}}x)}{\Delta_K(x^n)}$.

*where $V_K$ is the complexified volume of the knot complement, and $\Psi_K^{\sigma_1}(h)$ is the formal power series*

$$\Psi_K^{\sigma_1}(h) \in \overline{\mathbb{Q}}[[h]]$$

*associated to the geometric branch $x^{\sigma_1}(y)$ of the A-polynomial curve at $y = -1$.*

**Remark 0.2.** In comparison with the volume conjecture for colored Jones polynomials:

- The exponential term is still $e^{\frac{V_K}{2\pi} n}$.
- The power of $n$ after the exponential factor is $\frac{1}{2}$ here, which is different from $\frac{3}{2}$ of the asymptotic expansion of colored Jones polynomials.
- In the examples we have considered, the perturbative part $\Psi_K^{\sigma_1}(h)$ is the same as $\Phi_K^{\sigma_1}(h)$ up to sign.

**Example 0.3.** In case of the $\mathbf{4}_1$ knot,

$$F_{\mathbf{4}_1}(x,q) = -x - 3x^2 - (q^{-1} + 6 + q)x^3 - (2q^{-2} + 3q^{-1} + 11 + 3q + 2q^2)x^4 - \cdots,$$

and more explicitly,

$$F_{\mathbf{4}_1,n}(q) = -\sum_{0 \le i \le j \le n-1} \begin{bmatrix} j+i \\ 2i \end{bmatrix}.$$

By studying the asymptotics of $F_{\mathbf{4}_1,n}(e^{\frac{2\pi i}{n}})$, (4) can be numerically verified, with

$$\Psi_{\mathbf{4}_1}^{\sigma_1}(h) = -\frac{1}{\sqrt[4]{3}}\left(1 + \frac{11}{72\sqrt{-3}}h + \frac{697}{2(72\sqrt{-3})^2}h^2 + \frac{724351}{30(72\sqrt{-3})^3}h^3 + \cdots\right).$$

Note, this is exactly $-\Phi_{\mathbf{4}_1}^{\sigma_1}(h)$.
In fact, in this case, the exponential part of the volume conjecture can be proved analytically:

**Theorem 0.4.** *We have*

$$\lim_{n\to\infty} \frac{\log F_{\mathbf{4}_1,n}(e^{\frac{2\pi i}{n}})}{n} = \frac{V_{\mathbf{4}_1}}{2\pi}.$$

*Proof.* Setting $k = j - i$ and simplifying the expression, we need to show that

$$\lim_{n\to\infty} \frac{1}{n} \log \left( \sum_{\substack{0 \le i \le \lfloor \frac{n-1}{2} \rfloor \\ 0 \le k \le n-1-2i}} \exp \sum_{l=1}^{2i} \left( \log\sin\left(\pi\frac{k+l}{n}\right) - \log\sin\left(\pi\frac{l}{n}\right)\right)\right) = \frac{1}{\pi} D(e^{\frac{2\pi i}{6}}),$$

where $D$ is the Bloch-Wigner function; $D(e^{\frac{2\pi i}{6}})$ is the hyperbolic volume of the regular ideal tetrahedron. Since the outside summation is a summation of positive numbers over $\sim \frac{n^2}{4}$ pairs $(i,k)$, the left-hand side is equal to

$$\lim_{n\to\infty} \frac{1}{n} \log \left( \max_{\substack{0 \le i \le \lfloor \frac{n-1}{2} \rfloor \\ 0 \le k \le n-1-2i}} \exp \sum_{l=1}^{2i} \left( \log\sin\left(\pi\frac{k+l}{n}\right) - \log\sin\left(\pi\frac{l}{n}\right)\right)\right)$$

$$= \frac{1}{\pi} \lim_{n\to\infty} \max_{\substack{0 \le i \le \lfloor \frac{n-1}{2} \rfloor \\ 0 \le k \le n-1-2i}} \frac{\pi}{n} \sum_{l=1}^{2i} \left( \log\sin\left(\pi\frac{k+l}{n}\right) - \log\sin\left(\pi\frac{l}{n}\right)\right).$$

Using Euler-Maclaurin formula, we can replace the summation into an integral, and setting $\theta = \pi\frac{2i}{n}$ and $\theta' = \pi\frac{k}{n}$, the above expression becomes

$$\begin{aligned}&\frac{1}{\pi}\max_{\substack{0\le\theta,\theta',\\ \theta+\theta'\le\pi}}\left(-(\Lambda(\theta+\theta')-\Lambda(\theta'))+\Lambda(\theta)\right)\\ &=\frac{1}{\pi}\max_{\substack{0\le\theta,\theta',\\ \theta+\theta'\le\pi}}\left(\Lambda(\theta)+\Lambda(\theta')+\Lambda(\pi-\theta-\theta')\right),\end{aligned}$$

where $\Lambda(\theta) := -\int_0^\theta \log|2\sin(x)|dx$ is the Lobachevsky function. Since $\Lambda(\theta)+\Lambda(\theta')+\Lambda(\pi-\theta-\theta')$ is exactly the hyperbolic volume of the ideal tetrahedron with dihedral angles $\theta, \theta', \pi-\theta-\theta'$, which is maximized for the regular ideal tetrahedron, we conclude that this is equal to $\frac{1}{\pi}D(e^{\frac{2\pi i}{6}})$. $\square$

When $K$ is not hyperbolic, the right-hand side of (4) is in general a trans-series, consisting of contributions of flat connections whose real part of classical action is 0. For instance,

$$F_{\mathbf{3}_1,n}(e^{\frac{2\pi i}{n}}) \underset{n\to\infty}{\sim} e^{\frac{2\pi i}{24}n}n^{\frac{1}{2}}\Psi_{\mathbf{3}_1}^{\sigma_1}\left(\frac{2\pi i}{n}\right)+\Psi_{\mathbf{3}_1}^{\sigma_a}\left(\frac{2\pi i}{n}\right)+e^{-\frac{2\pi i}{3}n}\Psi_{\mathbf{3}_1}^{\sigma_b}\left(\frac{2\pi i}{n}\right)$$

where

$$\Psi_{\mathbf{3}_1}^{\sigma_1}(h) = e^{-\frac{2\pi i}{8}}\left(1-\frac{23}{2^3\cdot 3}h+\frac{529}{(2^3\cdot 3)^2}\frac{h^2}{2!}-\frac{12167}{(2^3\cdot 3)^3}\frac{h^3}{3!}+\frac{279841}{(2^3\cdot 3)^4}\frac{h^4}{4!}+\cdots\right) = e^{-\frac{2\pi i}{8}}q^{-\frac{23}{24}},$$

$$\Psi_{\mathbf{3}_1}^{\sigma_a}(h) = \frac{i}{\sqrt{3}}\left(1-\frac{2}{3}h+\frac{8}{3^2}\frac{h^2}{2!}+\frac{22}{3^3}\frac{h^3}{3!}+\frac{1136}{3^4}\frac{h^4}{4!}+\cdots\right),$$

$$\Psi_{\mathbf{3}_1}^{\sigma_b}(h) = \frac{i}{2\sqrt{3}}\left(1-\frac{11}{2^2\cdot 3}h+\frac{122}{(2^2\cdot 3)^2}\frac{h^2}{2!}-\frac{1358}{(2^2\cdot 3)^3}\frac{h^3}{3!}+\frac{15176}{(2^2\cdot 3)^4}\frac{h^4}{4!}+\cdots\right).$$

Note, $\Psi_{\mathbf{3}_1}^{\sigma_1}(h) = \Phi_{\mathbf{3}_1}^{\sigma_1}(h)$.

- Does $F_{K,n}(q)$ have Melvin-Morton-Rozansky-like expansion? Unlike $J_{K,n=\frac{u}{h}}(e^h)$, whose coefficients (as a power series in $h$) are polynomials in $u$, the coefficients of $F_{K,n=\frac{v}{h}}(e^h)$ (as a power series in $h$) grow exponentially in $v$. Therefore, there is no naive analogue of Melvin-Morton-Rozansky expansion for $F_{K,n}(q)$. Still, it is a curious question whether we can obtain the expansion (3) as some asymptotic expansion of $F_{K,n}(q)$.

**Question 0.5.**

$$F_{K,n}(e^h) \overset{?}{\sim} J_K(e^{nh}, e^h)$$

in some asymptotic expansion?

- The perturbative expansion of $F_K(x, e^h)$ is the Melvin-Morton-Rozansky expansion (2) of the colored Jones polynomials

$$F_K(x,e^h) = \frac{1}{\Delta_K(x)}+\frac{P_1(x)}{\Delta_K(x)^3}\frac{h}{1!}+\frac{P_2(x)}{\Delta_K(x)^5}\frac{h^2}{2!}+\cdots.$$

There are a few ways to further specialize this series.

  – The first is to set $x = 1$:

$$F_K(1,e^h) = -2J_K(-1,e^h).$$

– The second is to take residues of $\frac{1}{x}F_K(x,e^h)$ at some root of $\Delta_K(x)=0$. In this way, we obtain some perturbative series that appear in the asymptotics of $F_{K,n}(q)$.

For instance, the series $\Psi^{\sigma_a}_{\mathbf{3}_1}(h)$ that we saw earlier can be obtained as a residue:

$$\Psi^{\sigma_a}_{\mathbf{3}_1}(h) = -\mathrm{Res}_{x=e^{\frac{2\pi i}{6}}}\left(\frac{1}{x}F_{\mathbf{3}_1}(x,e^h)\right) = \mathrm{Res}_{x=e^{-\frac{2\pi i}{6}}}\left(\frac{1}{x}F_{\mathbf{3}_1}(x,e^h)\right).$$

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