LECTURES ON QUANTUM TOPOLOGY

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Abstract. These are **preliminary** and **incomplete** lecture notes for a Spring 2024 course at Harvard University. The most up-to-date version of these lecture notes can be found at https://categorical.center/Lectures_on_Quantum_Topology.pdf.

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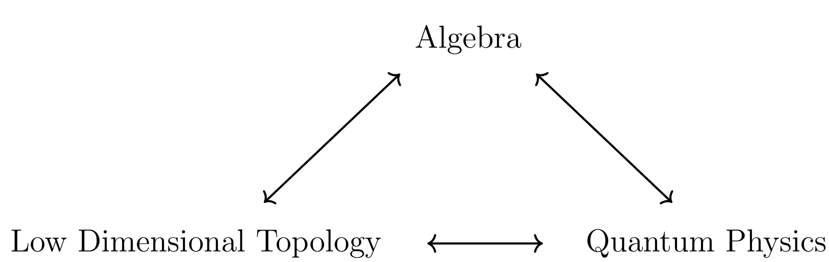
1. Lecture 1 (Tue Jan 23, 2024)

1.1. **What is Quantum Topology?** What is quantum topology? To put it in one sentence, it can probably be described as a branch of low-dimensional topology informed by Chern-Simons theory and its generalizations.

The advent of quantum topology can probably be traced back to the discovery of Jones polynomial [Jon85], Witten's interpretation of Jones polynomial in terms of Chern-Simons theory [Wit89], discovery of quantum groups [Dri85, Jim85] and the mathematically precise definition of Witten's 3-manifold invariants by Reshetikhin and Turaev [RT91] using quantum groups, axiomatization of TQFTs by Segal [Seg88] and Atiyah [Ati88], among many others.

1.1.1. *Low-dimensional topology, algebra, and physics.* One prominent feature of quantum topology is the close interaction among low-dimensional topology, algebra, and physics.



In quantum topology, we study invariants of knots, 3- and 4-manifolds that can be constructed out of interesting algebras (e.g. quantum groups) or categories (e.g. modular tensor categories). Those invariants, in turn, give us ways to think about the algebras and categories geometrically.

Those constructions are often motivated from physics, in which case there is a "quantum parameter" q. In the "classical limit" $q \to 1$, one recovers the corresponding "classical invariant" based on symplectic geometry.

1.1.2. *q-mathematics.* Another distinguishing feature of quantum topology is the omnipresence of q-analogs. The following quote is from Preface of the book "Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations" by Etingof, Frenkel, and Kirillov, Jr. [EFK98]:

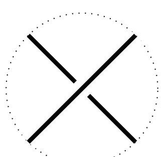
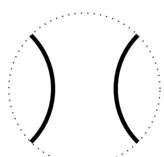
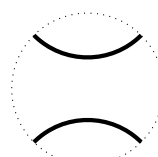
> "By that time, all three of us had already been severely afflicted with the "q-disease", a dangerous mathematical illness whose earliest victim was Euler, but which was first diagnosed by Richard Askey. Mathematicians working in practically every field, be it algebra, geometry, analysis, differential equations – you name it – are vulnerable to its addictive charm. The first symptom of the q-disease is that one day you realize that most of the results obtained or acquired during your mathematical life admit a q-deformation. The second stage is indicated by the idea that the q-case is much more interesting than the classical one. "

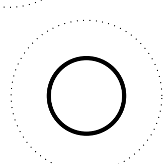
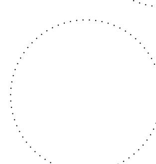
We – quantum topologists – fully embrace the q-disease. Indeed, we will encounter many q-analogs throughout this course.

1.2. **Historical overview.**

1.2.1. *Jones polynomial and skein relations.* Jones polynomial was first discovered from the study of von Neumann algebras [Jon85], but it was later reformulated by Kauffman in terms of skein relations. We will use this skein-theoretic approach.

Definition 1. The *Kauffman bracket* $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$ of a framed link L is the Laurent polynomial in A with integer coefficients that can be characterized by the following *skein relations* (all drawn in blackboard framing):

 $= A$  $+ \; A^{-1}$  ,

 $= (-A^2 - A^{-2})$  ,

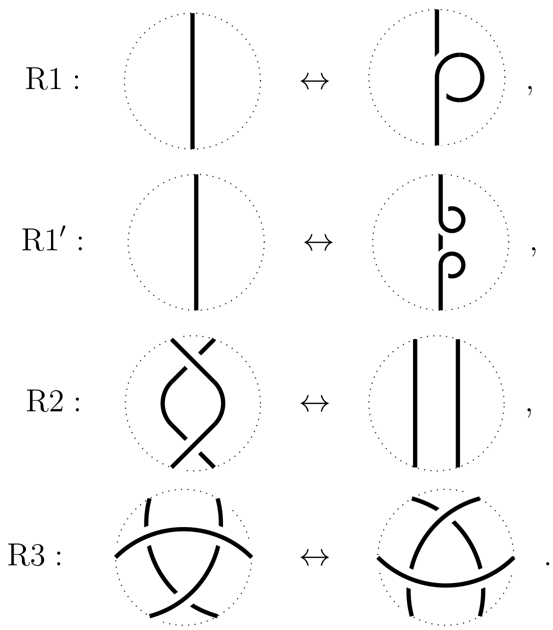
$$\langle \emptyset \rangle = 1.$$

Theorem 1. *The Kauffman bracket polynomial is well-defined.*

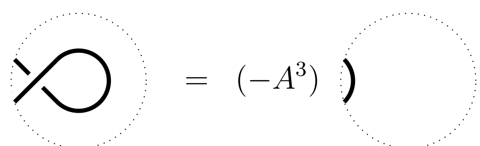
Before proving this theorem, let's recall the following old theorem by Reidemeister (and also independently by Alexander and Briggs):

Theorem 2 (Reidemeister moves)**.** *(1) Two link diagrams represent the same link iff they are related by a sequence of Reidemeister moves (R1, R2, and R3).*

(2) *Two link diagrams represent the same* framed *links in blackboard framing iff they are related by a sequence of framed Reidemeister moves (R1', R2, and R3).*



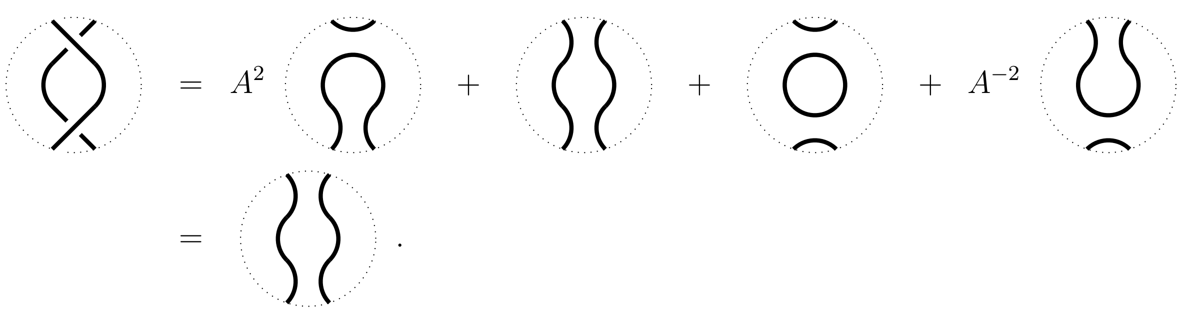
Proposition 1. *Under the R1 move (i.e. under the change of framing),*



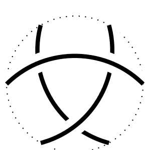
Proof. Easy exercise. □

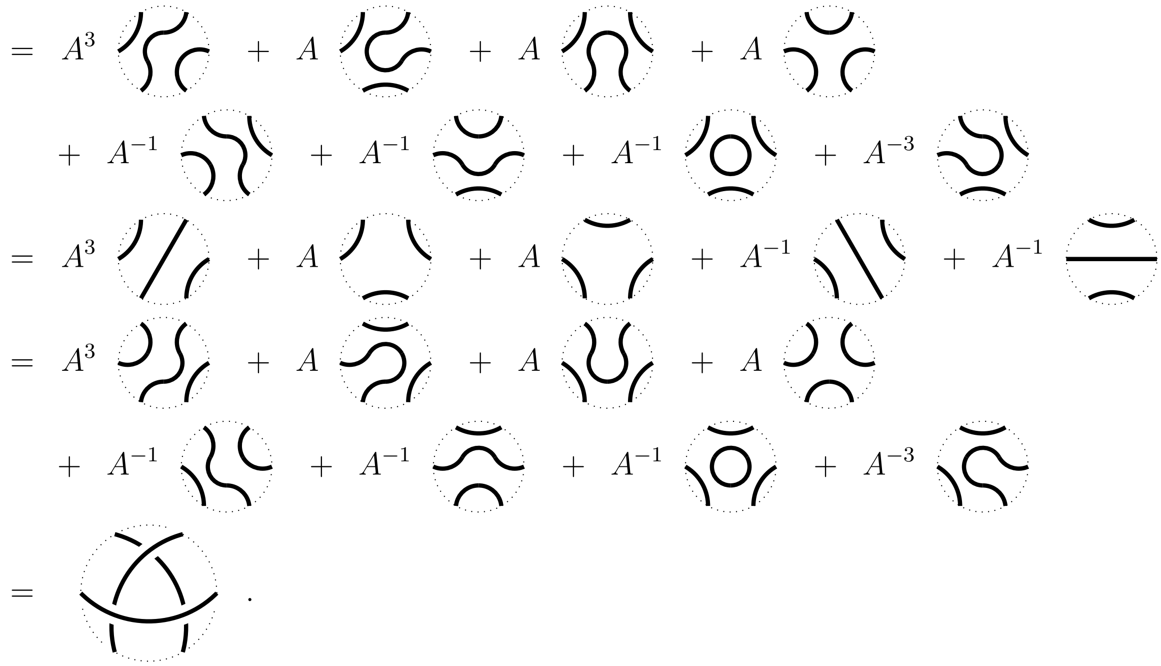
proof of Theorem 1. Because the framed first Reidemeister move (R1') and the second and the third Reidemeister moves (R2 and R3 moves) generate isotopies of framed links, it is enough to show that the Kauffman bracket polynomial is preserved under the R1', R2 and R3 moves. Invariance under R1' follows from Proposition 1.

It is preserved under the R2 move, because



Likewise, it is preserved under the R3 move, because





∎

Definition 2. For a framed, oriented link $L \subset S^3$, the *writhe* of L, denoted $w(L)$, is the self-linking number of L. It is equal to the signed number of crossings of its diagram in blackboard framing.

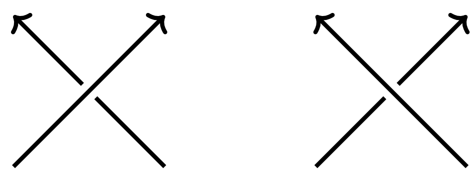


FIGURE 1. Positive (left) and negative (right) crossings

Definition 3. The *Jones polynomial* $J_L(q) \in \mathbb{Z}[q, q^{-1}]$ of an oriented, unframed link L is the Laurent polynomial in q with integer coefficients defined by

$$J_L(q = -A^2) \;:=\; (-A^3)^{-w(L)} \langle L \rangle,$$

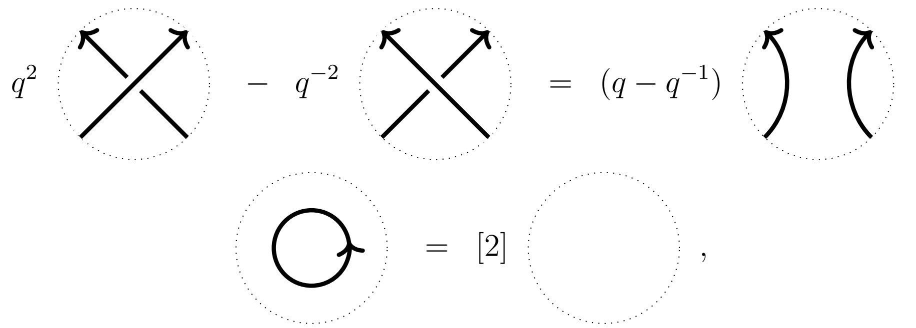
where $w(L)$ and $\langle L \rangle$ are computed in some diagram of L.

Theorem 3. *The Jones polynomial is well-defined.*

Proof. First of all, thanks to Proposition 1, it is clear that the Jones polynomial is invariant under all three Reidemeister moves (including R1) and hence is independent of the choice of link diagram.

Moreover, all the monomials of $\langle L \rangle$ have exponents whose parity is the same as the number of crossings in the diagram. It follows that $(-A^3)^{w(L)} \langle L \rangle$ has only even exponents, meaning $J_L(q = -A^2)$ is indeed a Laurent polynomial in q with integer coefficients. and this is true because ∎

Remark 1. The Jones polynomial can be equivalently defined in terms of the following skein relation:



$$J_\emptyset(q) \; = \; 1 \, ,$$

where [2] denotes the *quantum* 2:

$$[2] := \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}.$$

More generally, *quantum* n is defined as

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Notice that, in the *classical limit* $q \to 1$, the Jones polynomial doesn't see the crossings, and

$$J_L(q = 1) \; = \; 2^s$$

for an s-component link. As we will see later, this reflects the fact that the Jones polynomial is really an invariant of link "colored" by V_2, the standard 2-dimensional representation of $SU(2)$.

Remark 2. The Jones polynomial can distinguish many knots, and it is an open question whether the Jones polynomial detects the unknot. However, the Jones polynomial doesn't change under an operation called *mutation*, so, for instance, the Conway knot and the Kinoshita-Terasaka knot, which are mutants, have the same Jones polynomial.

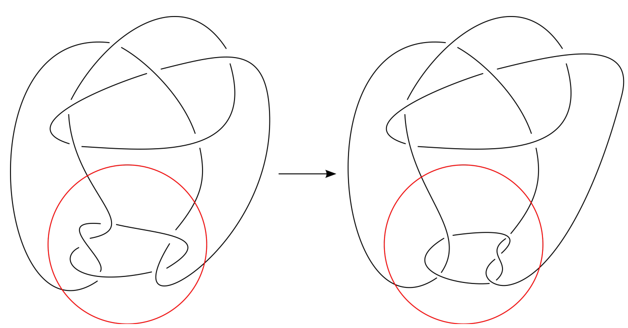
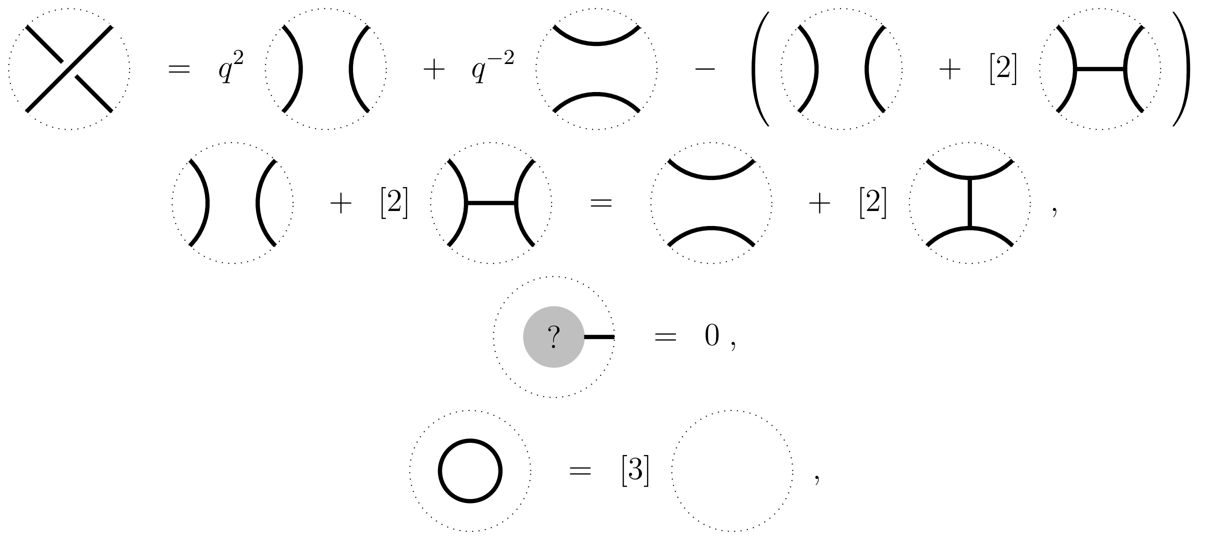


FIGURE 2. The Kinoshita-Terasaka knot (left) and the Conway knot (right). [Figure taken from Wikipedia]

Exercise 1. For any framed, unoriented spatial web (i.e. generalization of links where we allow trivalent vertices) K, define the $SO(3)$-polynomial $J_K^{SO(3)}(q) \in \mathbb{Z}[q, q^{-1}]$ using the following skein relations:



$$J_{\emptyset}^{SO(3)}(q) = 1.$$

Show that

(1) $J_K^{SO(3)}(q)$ is well-defined.

(2) The classical limit $J'^{SO(3)}_K(q=1)$ of the rescaled polynomial $J'^{SO(3)}_K(q) := [2]^{\frac{t}{2}} J_K^{SO(3)}(q)$, where t is the number of trivalent vertices of K, is equal to the number of Tait colorings (i.e. 3-colorings of the edges of K in such a way that every trivalent vertex meets all three colors) of K.

1.2.2. *Witten's interpretation of Jones polynomial in terms of Chern-Simons theory.* In [Wit89], Witten gave a physical interpretation of the Jones polynomial: the Jones polynomial is the expectation value of the Wilson line defect in Chern-Simons theory.

Witten's reformulation of Jones polynomial has at least 2 major advantages compared to the original formulation:

- It is manifestly 3-dimensional (i.e. the definition doesn't use any knot diagram at all), and
- it can be naturally extended to links L in any other 3-manifold Y.

The invariant $Z_{Y,L}$ of the pair (Y, L) was soon made mathematically rigorous by Reshetikhin and Turaev [RT91] using quantum groups and is now commonly known as the *Witten-Reshetikhin-Tureav invariant*.

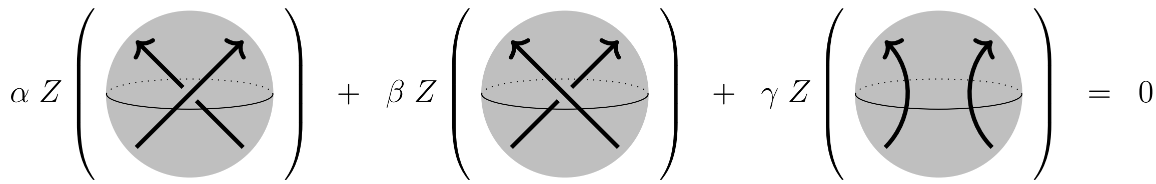
Chern-Simons theory, which is a 3d TQFT, also makes it clear why there should be such a skein relation. A 3d TQFT Z assigns a vector space $Z(\Sigma)$ to a closed surface Σ and a vector $Z(Y) \in Z(\partial Y)$ to a 3-manifold Y with boundary ∂Y. So, given a tangle T in a ball B^3, Chern-Simons theory assigns a vector $Z(B^3, T)$ in the vector space $Z(S^2, \partial T)$ assigned to S^2 with marked points.

From physics, Witten derived that (as long as the level k is big enough) the dimension of the vector space $Z(S^2, \{p_1, p_2, p_3, p_4\})$ assigned to the 2-sphere with 4 marked points is equal to

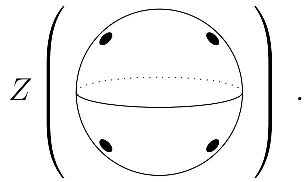
$$\dim \mathrm{Inv}(V_2 \otimes V_2 \otimes V_2 \otimes V_2) = 2,$$

where Inv denotes the $SU(2)$-invariant subspace.

Any 2-dimensional vector space has a marvelous property that any 3 vectors in that vector space satisfies a non-trivial linear relation. Therefore, we should have



for some $\alpha, \beta, \gamma \in \mathbb{C}$ (not all of them zero) as vectors in



This is the origin of the 3-term skein relation.

2. Lecture 2 (Thu Jan 25, 2024)

2.1. **Historical overview (cont.)**

2.1.1. *TQFTs.* Soon after Witten's work on TQFTs, Segal and Atiyah mathematically formalized the notion of TQFTs. Mathematically, in the simplest form, an n-dimensional TQFT Z is a symmetric monoidal functor from $\mathrm{Bord}_{n,n-1}$ (the category whose objects are closed $(n-1)$-manifolds and whose morphism are n-dimensional bordisms (up to homotopy) between them) to the category of vector spaces and linear maps:

Definition 4 ([Ati88])**.** An n-dimensional *topological quantum field theory (TQFT)* Z is a symmetric monoidal functor

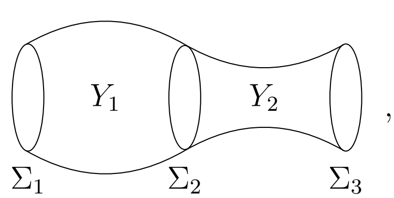
$$Z : \mathrm{Bord}_{n,n-1} \to \mathrm{Vect}_{\mathbb{C}}.$$

In other words, it is a functor assigning

(1) a finite-dimensional complex vector space $Z(\Sigma)$ to each compact oriented smooth $(n-1)$-manifold Σ, and
(2) a vector $Z(Y) \in Z(\Sigma)$ for each compact oriented n-manifold Y with boundary Σ.

This functor should satisfy the following axioms:

(A1) (Involutory) $Z(\Sigma^*) = Z(\Sigma)^*$, where Σ^* denotes Σ with opposite orientation, and $Z(\Sigma)^*$ is the dual space.
(A2) (Multiplicativity) $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$.
(A3) (Associativity) For a composite bordism $Y = Y_1 \cup_{\Sigma_2} Y_2$



$Z(Y) = Z(Y_2) \circ Z(Y_1) \in \mathrm{Hom}(Z(\Sigma_1), Z(\Sigma_3))$.
(A4) $Z(\emptyset) = \mathbb{C}$.
(A5) $Z(\Sigma \times I) = \mathrm{id}_{Z(\Sigma)}$.

One of the main goal of quantum topology is to produce a lot of interesting TQFTs with potential applications to low-dimensional topology. In a slightly different direction, classification of TQFTs itself is an interesting problem in its own right. The classification of *fully extended* TQFTs was conjectured by Baez and Dolan [BD95], known as the *cobordism hypothesis*, and was proved by Lurie [Lur09]. While we will not go to this direction in this course, let me just mention an old result by Dijkgraaf on classification of 2d TQFTs, as it might be useful later:

Theorem 4 ([Dij89])**.** *The category* $\mathrm{Bord}_{2,1}$ *is freely generated, as a symmetric monoidal category, by a commutative Frobenius object* S^1*. In other words, the data of a 2d TQFT* $Z : \mathrm{Bord}_{2,1} \to \mathrm{Vect}$ *is equivalent to the data of a commutative Frobenius algebra* $Z(S^1) \in \mathrm{Vect}$*.*

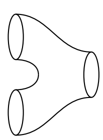
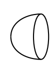
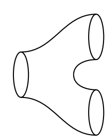
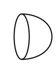
Definition 5. A *Frobenius algebra* A is a finite-dimensional unital associative algebra (A, μ, η) equipped with a linear form ("counit", sometimes called "trace") $\epsilon : A \to \Bbbk$ such that $\epsilon \circ \mu : A \otimes A \to \Bbbk$ is a non-degenerate pairing (i.e. induces an isomorphism $A \to A^*$).

Equivalently, a Frobenius algebra is a tuple $(A, \mu, \eta, \Delta, \epsilon)$ such that

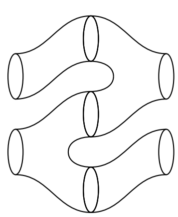
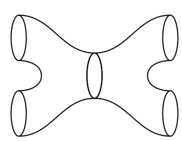
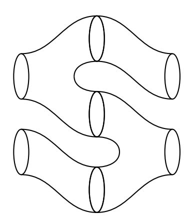
(1) (A, μ, η) is an algebra with unit η,
(2) (A, Δ, ϵ) is a coalgebra with counit ϵ, and
(3) the Frobenius relation
$$(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = \Delta \circ \mu = (\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id})$$
is satisfied.

We say that a Frobenius algebra is *commutative* if the associated algebra is commutative (or, equivalently, the associated coalgebra is cocommutative).

Diagrammatically, reading from left to right, we can draw μ, η, Δ, and ϵ as

μ =  , η =  , Δ =  , ϵ =  .

Then, the Frobenius relation can be expressed as

 =  =  .

Examples of Frobenius algebras include matrix rings, group rings, ring of characters of a representation, cohomology rings, etc.

Example 1 ($\mathbb{CP}^{N-1}$-model)**.** In $\mathbb{CP}^{N-1}$-model, we assign
$$Z(S^1) = H^*(\mathbb{CP}^{N-1}) \cong \frac{\mathbb{C}[x]}{(x^N = 0)}.$$
The counit is given by the linear map
$$\begin{aligned} \epsilon : Z(S^1) &\to \mathbb{C} \\ x^k &\mapsto \begin{cases} 1 & \text{if } k = N-1 \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

and the coproduct is given by

$$\begin{aligned}\Delta : Z(S^1) &\to Z(S^1) \otimes Z(S^1)\\ x^k &\mapsto \sum_{0 \le j \le N-1-k} x^{k+j} \otimes x^{N-1-j}.\end{aligned}$$

This is the 2d TQFT behind the $\mathfrak{sl}_N$ link homologies, which we might cover later in the course.

The $\mathbb{CP}^{N-1}$-model admits a nice deformation, namely to use $U(N)$-equivariant cohomology instead of ordinary cohomology. That is, instead of setting $x^N = 0$, we set some generic monic polynomial of degree N to be 0:

$$Z'(S^1) = \frac{R[x]}{((x-x_1)\cdots(x-x_N)=0)},$$

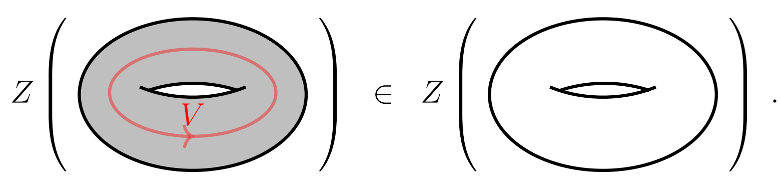
where $R := \mathbb{C}[x_1, \cdots, x_N]^{S_N}$. This is the 2d TQFT behind the equivariant $\mathfrak{sl}_N$ link homologies. The counit remains the same as in the undeformed case.

Exercise 2. Compute $Z(\Sigma_g)$ in the $\mathbb{CP}^{N-1}$-model and its deformation, where Σ_g denotes the closed oriented surface of genus g.

2.1.2. *Quantum groups and the work of Reshetikhin and Turaev.* Witten's generalization of Jones polynomial to 3-manifolds was made mathematically precise by Reshetikhin and Turaev [RT91] and is now commonly known as the Witten-Reshetikhin-Turaev (WRT) invariant.

Reshetikhin and Turaev's construction, namely construction of link invariants from representations of a *quantum group*, or more generally a *ribbon category*, and a 3d TQFT from representations of a quantum group at a root of unity, or more generally a (semisimple) *modular tensor category* (MTC), will be covered in detail in this course, mostly following [BK01] or [Tur94].

The basic idea is that, $\mathcal{C}$ is the category of line operators, and the vector space associated to the torus T^2 is finite dimensional and has basis labeled by simple objects of $\mathcal{C}$:



Theorem 5 ([RT91, Tur94]). *Given an MTC $\mathcal{C}$, one can construct an invariant of 3-manifolds with colored links inside them. Moreover, this can be extended to a 3d TQFT.*

2.1.3. *Potential topics.* Let me conclude this overview by listing tentative topics to be covered in this course.

For the first half of this course, the current plan is to cover the following topics:

- Finite type (Vassiliev) invariants and perturbative Chern-Simons theory
- Basics of Hopf algebras, quantum groups and their representation theory
- Reshetikhin-Turaev construction (ribbon categories and link invariants, modular tensor categories and 3d TQFTs)

The later half of this course will cover more recent topics, more relevant to current research:

- Stated skein algebras and modules, and quantum trace maps
- Non-semisimple invariants and TQFTs (Costantino-Geer-Patureau invariants)

We may add or remove topics from this list depending on how much time we have left as the course proceeds.

2.2. **Temperley-Lieb-Jones algebroids.** Before delving into ribbon categories, let's first study a concrete example, namely Temperley-Lieb algebroids. We follow [Wan10, Ch. 1] in this subsection. (See also [Tur94, Ch. 12])

2.2.1. *Temperley-Lieb algebroids.*

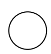
Definition 6. Let $\Bbbk$ be a field. An $\Bbbk$-*algebroid* is a small $\Bbbk$-linear category (i.e. Hom sets are vector spaces, and the composition maps are bilinear).

For a $\Bbbk$-algebroid Λ, we will sometimes denote its set of objects as Λ^0, and $\mathrm{Hom}(x, y)$ as ${}_x\Lambda_y$.

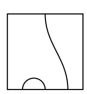
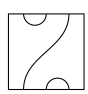
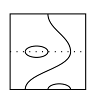
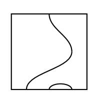
Proposition 2. *For any objects x, y in a $\Bbbk$-algebroid Λ, ${}_x\Lambda_x$ is a $\Bbbk$-algebra, and ${}_x\Lambda_y$ is a ${}_y\Lambda_y - {}_x\Lambda_x$-bimodule.*

Definition 7. The *Temperley-Lieb (TL) algebroid* $\mathrm{TL}(A)$ is a $\mathbb{C}(A)$-algebroid defined as follows:

(1) An object of $\mathrm{TL}(A)$ is the unit interval I with a finite set of points in the interior of I, allowing the emptyset. Let $|x|$ denote the number of points in x, for $x \in \mathrm{TL}(A)^0$.
(2) The set of morphisms ${}_x\mathrm{TL}(A)_y$ is given by the vector space spanned by the isotopy classes of smooth, pairwise non-intersecting arcs and loops in the box $I \times I$ whose intersection with $I \times \{0\}$ is x and the intersection with $I \times \{1\}$ is y, modulo the relation

 $= d := -A^2 - A^{-2}.$

(3) Composition of morphism is given by vertical concatenation of boxes. For instance,

 $\circ$  $=$  $= d$  .

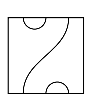
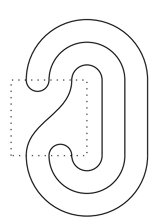
Remark 3. Note, all objects x of the same cardinality $|x|$ are isomorphic. We will denote the isomorphism class of objects x with $|x| = n$ by 1^n.

Definition 8. Given a natural number $n \in \mathbb{N}$, the *Temperley-Lieb algebra* $\mathrm{TL}_n(A)$ is the algebra $\mathrm{Hom}(1^n, 1^n)$ of the TL algebroid.

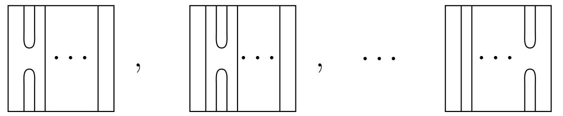
Definition 9. The *Markov trace* of $TL_n(A)$ is the algebra homomorphism

$$\mathrm{tr} : \mathrm{TL}_n(A) \to \mathbb{C}(A)$$

defined by the tracial closure: close up the top and the bottom ends by non-intersecting arcs connecting top to bottom, and then evaluate it to d^m, where m is the number of circles after the closure. For instance,

tr  $=$  $= d.$

Let $\{U_i\}_{1\le i\le n-1}$ be the TL diagrams in $\mathrm{TL}_n(A)$ shown below:



They generate $\mathrm{TL}_n(A)$ as a unital algebra.

Proposition 3. *The set of all loopless TL diagrams forms a basis of* $\mathrm{TL}_n(A)$ *as a vector space. In particular,*

$$\dim \mathrm{TL}_n(A) = \frac{1}{n+1}\binom{2n}{n} =: c_n,$$

the n-th Catalan number.

Proposition 4. *The generators* $\{U_i\}_{1\le i\le n-1}$ *of* $\mathrm{TL}_n(A)$ *satisfy the following relations:*

(1) $U_i^2 = d\cdot U_i$,
(2) $U_iU_{i\pm1}U_i = U_i$,
(3) $U_iU_j = U_jU_i$ *if* $|i-j|\ge 2$

In fact, these generate all the relations between U_i's.

Theorem 6. $\mathrm{TL}_n(A)$ *is isomorphic to a direct sum of matrix algebras over* $\mathbb{C}(A)$.

3. Lecture 3 (Tue Jan 30, 2024)

3.1. Temperley-Lieb-Jones algebroids (cont.)

3.1.1. *Jones' braid group representation.* Recall that the n-strand braid group B_n has a presentation

$$B_n = \langle \{\sigma_i\}_{1\le i\le n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|\ge 2\rangle.$$

Proposition 5. *The Kauffman bracket*

$$\langle , \rangle : \mathbb{C}(A)[B_n] \to \mathrm{TL}_n(A)$$
$$\sigma_i \mapsto A\cdot 1 + A^{-1}U_i$$

induces a surjective algebra homomorphism.

Proof. It is straightforward to check that this map respects the braid relations. □

Remark 4. Since $\mathrm{TL}_n(A)$ is isomorphic to a direct sum of matrix algebras, the Kauffman bracket yields a representation of B_n, called the *Jones representation.*

3.1.2. *Jones-Wenzl projectors.* Recall that the n-th Chebyshev polynomial $\Delta_n(d)$ is defined inductively by $\Delta_0(d) = 1$, $\Delta_1(d) = d$, $\Delta_{n+1}(d) = d\Delta_n(d) - \Delta_{n-1}(d)$. Note,

$$\Delta_n([2]) = [n+1].$$

Theorem 7 (Jones-Wenzl projectors)**.** $\mathrm{TL}_n(A)$ *contains a unique element p_n characterized by:*

(1) $p_n \ne 0$.
(2) $p_n^2 = p_n$.
(3) $U_ip_n = p_nU_i = 0$ *for all* $1\le i\le n-1$.

Moreover, p_n can be written as $p_n = 1 + U$, where U is a linear combination of non-trivial monomials of U_i's.

Proof. For uniqueness, suppose p_n exists and can be expressed as

$$p_n = c\,1 + U.$$

Then,

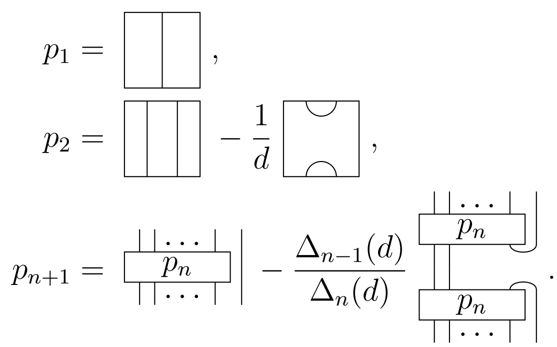
$$p_n^2 = p_n(c\,1 + U) = c\,p_n = c^2\,1 + c\,U,$$

so c must be 1. Suppose that $p_n = 1 + U$ and $p'_n = 1 + V$ both have the properties above. Then,

$$p'_n = (1 + U)p'_n = p_n p'_n = p_n(1 + V) = p_n.$$

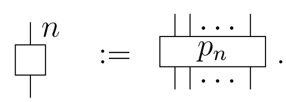
This proves uniqueness.

Existence can be shown inductively by



∎

For simplicity, we will sometimes use a strand labeled n for n parallel strands, and a box for the corresponding Jones-Wenzl projector:



Remark 5. Really, one should think of the Jones-Wenzl projector as the projector

$$p_n : V_2^{\otimes n} \to V_{n+1} \subset V_2^{\otimes n},$$

where V_2 denotes the standard 2-dimensional representation of $SU(2)$ and V_n denotes the n-dimensional irreducible representation.

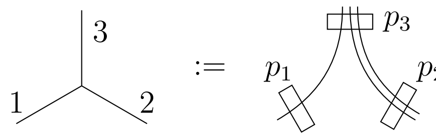
3.1.3. *Trivalent graphs and bases of morphism spaces.* Consider (planar) uni-trivalent graphs in the square $I \times I$, allowing loops and multi-edges, such that all the trivalent vertices are in the interior and all the univalent vertices are either in the bottom $I \times \{0\}$ or in the top $I \times \{1\}$. (The top and the bottom edges represent some objects of $\mathrm{TL}(A)$.)

Given such a uni-trivalent graph G, a *coloring* of G is an assignment of natural numbers to each edge of G such that all the edges with univalent vertices are colored by 1.

A coloring is called *admissible* if for every trivalent vertex, the three colors a, b, c adjacent to it satisfy

(1) $a + b + c$ is even.
(2) $a + b \geq c, \quad b + c \geq a, \quad c + a \geq b.$

Any admissibly colored uni-trivalent graph with bottom object x and top object y represents a morphism in ${}_x\mathrm{TL}(A)_y$, by the following rules of insertions of Jones-Wenzl projectors:



The following proposition generalizes Proposition 3.

Proposition 6. *Let x and y be two objects of* $\mathrm{TL}(A)$ *such that $|x|+|y|=2m$. Then*

(1) $\dim {}_x\mathrm{TL}(A)_y = \frac{1}{m+1}\binom{2m}{m}$.
(2) For any connected uni-trivalent tree G connecting x and y, the set of all admissible colorings of G forms a basis of ${}_x\mathrm{TL}(A)_y$.

3.1.4. *Temperley-Lieb-Jones algebroids.*

Definition 10. The *Temperley-Lieb-Jones (TLJ) algebroid* $\mathrm{TLJ}(A)$ is the $\mathbb{C}(A)$-algebroid, whose objects are objects of $\mathrm{TL}(A)$ but with natural number colors, and whose morhpisms between x and y are formal linear combinations of uni-trivalent graphs with admissible colorings compatible with the colored objects x, y.

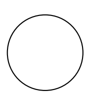
Both $\mathrm{TL}(A)$ and $\mathrm{TLJ}(A)$ have tensor products which is horizontal juxtaposition, with the empty object being the tensor unit. This makes them monoidal categories. Even better:

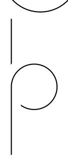
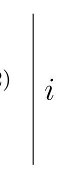
Theorem 8. $\mathrm{TL}(A)$ *and* $\mathrm{TLJ}(A)$ *are ribbon categories.*

We will study ribbon categories soon, so for now, let me just say informally that, a ribbon category is a monoidal category with a braiding, a twist, and a compatible duality (cups and caps).

Definition 11. Let L be a framed trivalent graph with an admissible coloring. Its Kauffman bracket is called the *colored Kauffman bracket* $\langle L\rangle$.

Proposition 7. *The colored Kauffman bracket satisfies*

(1)  $i \quad = [i+1] := \frac{q^{i+1}-q^{-i-1}}{q-q^{-1}} = \frac{(-A^2)^{i+1}-(-A^2)^{-i-1}}{(-A^2)-(-A^2)^{-1}}$,

(2)  $i \quad = (-1)^i A^{i(i+2)}$  i

Definition 12. The *n-colored Jones polynomial* of an oriented link L $J_{n,L}(q)$ is defined as

$$J_{n,L}(q=-A^2) = ((-1)^n A^{n(n+2)})^{-w(L)}\langle L\rangle_A,$$

where the colored Kauffman bracket is evaluated for L colored by n.

When $n=1$, this is the usual Jones polynomial.

Remark 6. Note, the unknot colored by p_n evaluates to $\Delta_n([2]) = [n+1]$.

Exercise 3. Derive the skein relations given in Exercise 1. (Hint: color the strands by the second Jones-Wenzl projector p_2 and evaluate in $\mathrm{TL}(A)$.)

4. Lecture 4 (Thu Feb 1, 2024)

4.1. **Ribbon categories.** We follow [Tur94, Ch. 1].

4.1.1. *Monoidal categories.* Recall that, for any category $\mathcal{C}$, the Cartesian square of $\mathcal{C}$ – $\mathcal{C} \times \mathcal{C}$ – is a category whose objects are pairs of objects of $\mathcal{C}$, morphisms are ordered pairs of morphisms, and the compositions are component-wise composition in $\mathcal{C}$.

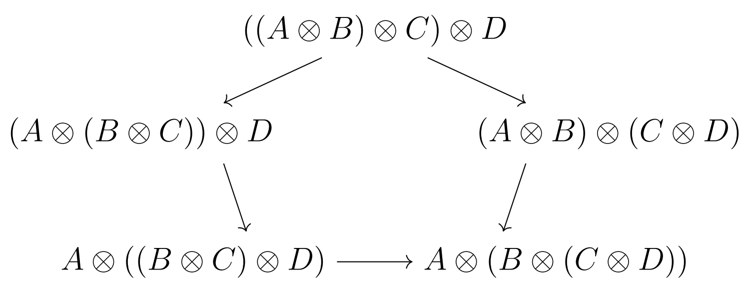
Definition 13. A category $\mathcal{C}$ is called *monoidal* if it is equipped with a functor (called the *tensor* or *monoidal product*)

$$\begin{aligned} \otimes : \mathcal{C} \times \mathcal{C} &\to \mathcal{C}, \\ (A, B) &\mapsto A \otimes B, \\ (f, g) &\mapsto f \otimes g, \end{aligned}$$

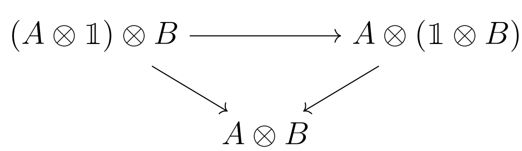
an object $\mathbb{1}$ (called the *unit* or *identity object*), and natural isomorphisms (respecively called *associator*, *left unitor* and *right unitor*)

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad A, B, C \in \operatorname{Ob} \mathcal{C},$$
$$\lambda_A : \mathbb{1} \otimes A \to A, \quad \rho_A : A \otimes \mathbb{1} \to A, \quad A \in \operatorname{Ob} \mathcal{C},$$

which satisfy two families of coherence conditions corresponding to commutative diagrams (respectively called the *pentagon* and the *triangle diagrams*)



and



,

and the equality

$$\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}.$$

A monoidal category is called *strict* if the natural isomorphisms α, λ, ρ are identities. It is known (MacLane's coherence theorem) that any monoidal category is equivalent to a strict monoidal category.

Example 2. The category of vector spaces, $\mathrm{Vect}_{\mathbb{k}}$, is a monoidal category.

4.1.2. *Braided categories.* Let $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the *exchange* functor, defined by exchanging the components:

$$\begin{aligned} \tau(A,B) &= (B,A), \quad A,B \in \mathrm{Ob}\,(\mathcal{C} \times \mathcal{C}) \\ \tau(f,g) &= (g,f). \end{aligned}$$

Define

$$\begin{aligned} \otimes^{\mathrm{op}} := \otimes \circ \tau : \mathcal{C} \otimes \mathcal{C} &\to \mathcal{C} \\ (A,B) &\mapsto B \otimes A, \\ (f,g) &\mapsto g \otimes f. \end{aligned}$$

Definition 14. A monoidal cateogry is called *braided* if it is equipped with a natural isomorphism (called *braiding*)

$$\beta : \otimes \to \otimes^{\mathrm{op}},$$

(i.e. a natural family of isomorphisms

$$\beta = \{\beta_{A,B} : A \otimes B \to B \otimes A\}_{A,B \in \mathrm{Ob}\,\mathcal{C}}$$

where naturality means that $(g \otimes f)\beta_{A,B} = \beta_{A',B'}(f \otimes g)$ for any $f : A \to A'$, $g : B \to B'$) such that the following diagrams (called *hexagon diagrams*) are commutative:

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \longrightarrow & (B \otimes C) \otimes A & \longrightarrow & B \otimes (C \otimes A) \\ \uparrow & & & & \uparrow \\ (A \otimes B) \otimes C & \longrightarrow & (B \otimes A \otimes C) & \longrightarrow & B \otimes (A \otimes C) \end{array}$$

and

$$\begin{array}{ccccc} (A \otimes B) \otimes C & \longrightarrow & C \otimes (A \otimes B) & \longrightarrow & (C \otimes A) \otimes B \\ \uparrow & & & & \uparrow \\ A \otimes (B \otimes C) & \longrightarrow & A \otimes (C \otimes B) & \longrightarrow & (A \otimes C) \otimes B \end{array} .$$

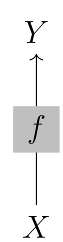
A braided category is called *symmetric* if the braiding satisfies

$$\beta_{A,B}^{-1} = \beta_{B,A}, \quad \forall A,B \in \mathrm{Ob}\,\mathcal{C}.$$

In this case, the braiding is called *symmetry* and is commonly denoted as σ.

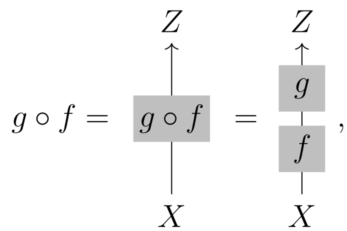
4.1.3. *String diagrams.* From now on, let $\mathcal{C}$ be a strict monoidal category. Then, it is often convenient to use the graphical notation of *string diagrams*.

A morphism $f : X \to Y$ in $\mathcal{C}$ will be depicted graphically as

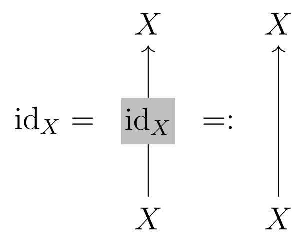
$f =:$ 

.

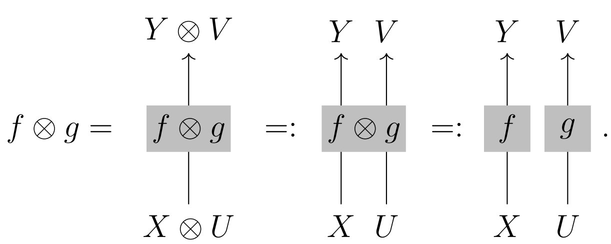
Composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ is described by the vertical concatenation



and the identity morphism id_X is drawn as a line:

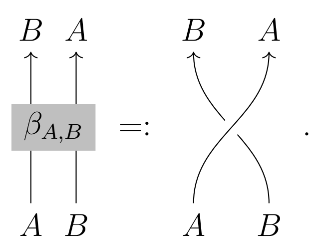


The tensor product can be drawn by the horizontal juxtaposition:

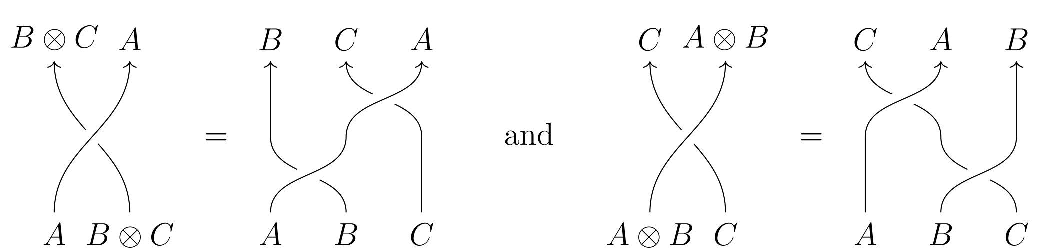


The identity object $\mathbb{1}$ is naturally associated to the empty graph.

It is natural to depict a braiding by



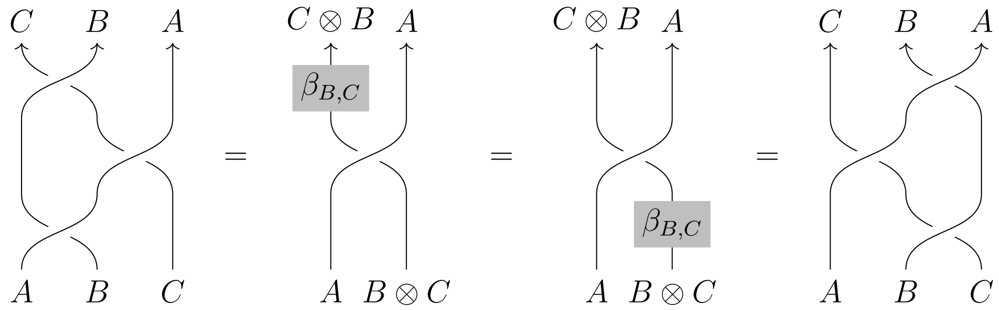
Then, the hexagon diagrams for the braiding becomes



Proposition 8. *Any braiding satisfies the Yang-Baxter identity:*

$$(\beta_{B,C} \otimes \mathrm{id}_A)(\mathrm{id}_B \otimes \beta_{A,C})(\beta_{A,B} \otimes \mathrm{id}_C) = (\mathrm{id}_C \otimes \beta_{A,B})(\beta_{A,C} \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \beta_{B,C}).$$

Proof.



□

4.1.4. *Ribbon categories.*

Definition 15. A monoidal category is called *right (resp. left) rigid* if for every object A, there exists an object A^* called the *right dual* (resp. an object ${}^*A$ called the *left dual*) and associated morphisms

$$\overleftarrow{\cup}_A : \mathbb{1} \to A \otimes A^*, \quad \overleftarrow{\cap}_A : A^* \otimes A \to \mathbb{1} \quad \text{(for right duals)}$$

$$\overrightarrow{\cup}_A : \mathbb{1} \to {}^*A \otimes A, \quad \overrightarrow{\cap}_A : A \otimes {}^*A \to \mathbb{1} \quad \text{(for left duals)}$$

satisfying the *zig-zag identities*:

$$(\mathrm{id}_A \otimes \overleftarrow{\cap}_A)(\overleftarrow{\cup}_A \otimes \mathrm{id}_A) = \mathrm{id}_A,$$

$$(\overleftarrow{\cap}_A \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A*} \otimes \overleftarrow{\cup}_A) = \mathrm{id}_{A^*}$$

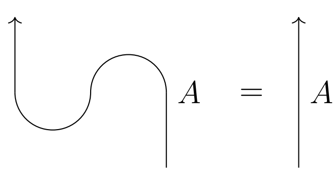
for right duals, and

$$(\mathrm{id}_{{}^*A} \otimes \overrightarrow{\cap}_A)(\overrightarrow{\cup}_A \otimes \mathrm{id}_{{}^*A}) = \mathrm{id}_{{}^*A},$$

$$(\overrightarrow{\cap}_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \overrightarrow{\cup}_A) = \mathrm{id}_A$$

for left duals. The data of (right or left) duals is called a (right or left) *duality* in the monoidal category.

Graphically, the zig-zag identities can be visualized as



for the first identity, and similarly for the other ones.

Definition 16. A *twist* (or *blance*) θ in a braided category $\mathcal{C}$ is a natural isomorphism of the identity functor $\mathrm{id}_{\mathcal{C}}$ to itself:

$$\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}},$$

(i.e. a natural family of isomorphisms

$$\theta = \{\theta_A : A \to A\}_{A \in \mathrm{Ob}\, \mathcal{C}}$$

where naturality means that $\theta_B f = f \theta_A$ for any $f : A \to B$) such that

$$\theta_{A \otimes B} = \beta_{B,A} \beta_{A,B} (\theta_A \otimes \theta_B).$$

By naturality of the braiding, we can write

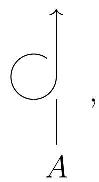
$$\theta_{A\otimes B} = \beta_{B,A}\beta_{A,B}(\theta_A \otimes \theta_B) = \beta_{B,A}(\theta_B \otimes \theta_A)\beta_{A,B} = (\theta_A \otimes \theta_B)\beta_{B,A}\beta_{A,B}.$$

Note also that $\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$, which follows from

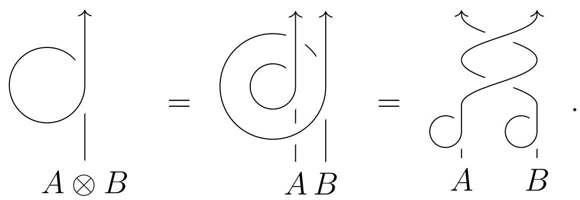
$$(\theta_{\mathbb{1}})^2 = (\theta_{\mathbb{1}} \otimes \mathrm{id}_{\mathbb{1}})(\mathrm{id}_{\mathbb{1}} \otimes \theta_{\mathbb{1}}) = \theta_{\mathbb{1}} \otimes \theta_{\mathbb{1}} = \theta_{\mathbb{1}}$$

and invertibility of $\theta_{\mathbb{1}}$.

Diagrammatically, the twist θ_A can be drawn as



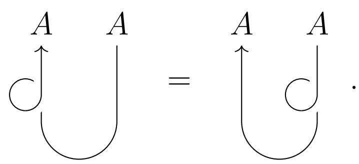
and the compatibility with braiding can be drawn as



Definition 17. A *ribbon category* is a braided category equipped with a twist θ, and a compatible (right) duality $(*, \overleftarrow{\cup}, \overleftarrow{\cap})$, where compatibility means that, for any object A,

$$(\theta_A \otimes \mathrm{id}_{A^*})\overleftarrow{\cup}_A = (\mathrm{id}_A \otimes \theta_{A^*})\,\overleftarrow{\cup}_A\,.$$

Graphically, the compatibility of the duality with the twist can be depicted as



Remark 7. Just to summarize,

modular categories $\subset$ ribbon categories $\subset$ braided categories $\subset$ monoidal categories.

We haven't defined *modular categories* yet, but they will appear later in the course, when we discuss 3-manifold invariants and 3d TQFTs.

5. Lecture 5 (Tue Feb 6, 2024)

5.1. **Ribbon categories (cont.)**

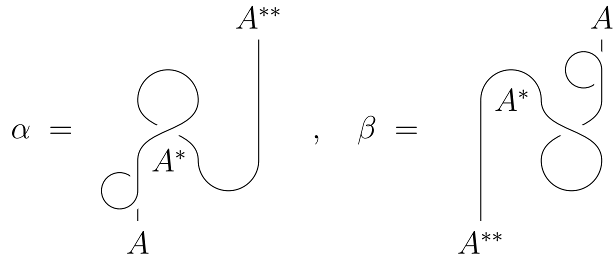
Proposition 9. *For any object A of a ribbon category, there is a natural, monoidal isomorphism $A \xrightarrow{\sim} A^{**}$. That is, ribbon categories are* pivotal, *and in particular, any right dual can also be thought of as a left dual.*

Proof. Consider the following morhpisms

$$\alpha = ((\overleftarrow{\cap}_A \circ \beta_{A,A^*}) \otimes \mathrm{id}_{A^{**}}) \circ (\theta_A \otimes \overleftarrow{\cup}_{A^*}) \;\in\; \mathrm{Hom}(A, A^{**}),$$

$$\beta = (\overleftarrow{\cap}_{A^*} \otimes \theta_A^{-1}) \circ (\mathrm{id}_{A^{**}} \otimes (\beta_{A^*,A}^{-1} \circ \overleftarrow{\cup}_A)) \;\in\; \mathrm{Hom}(A^{**}, A).$$

In terms of string diagrams,



We claim that $\beta \circ \alpha = \mathrm{id}_A$ and $\alpha \circ \beta = \mathrm{id}_{A^{**}}$ so that they provide isomorphisms between A and A^{**}. This can be shown by a sequence of elementary isotopies of the string diagrams, each of which give the same morhpism thanks to the naturality of braiding, twist, and the properties of the duality morphisms.

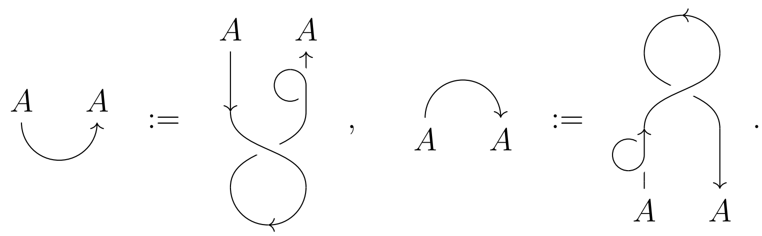
In a similar way, one can also show that this isomorphism is monoidal and natural. The details are left as an exercise.[1] □

Remark 8. Note, the twists in the definition of the isomorhpism $A \xrightarrow{\sim} A^{**}$ are necessary for monoidality.

It will be useful to define the following (left) duality morphisms, which are just β and α composed with the usual (right) duality morphisms:

$$\overrightarrow{\cup}_A := (\mathrm{id}_{A^*} \otimes \theta_A^{-1}) \circ \beta^{-1}_{A^*,A} \circ \overleftarrow{\cup}_A,$$
$$\overrightarrow{\cap}_A := \overleftarrow{\cap}_A \circ \beta_{A,A^*} \circ (\theta_A \otimes \mathrm{id}_{A^*}).$$

That is,



It follows from Proposition 9 that they satisfy the zig-zag identities for left duals.

Example 3. The categories $\mathrm{TL}(A)$ and $\mathrm{TLJ}(A)$ we saw earlier are ribbon categories, with obvious braiding, twist, and duality.

Example 4. Let R be a commutative ring with unit. Then, the category $\mathrm{Proj}(R)$ of finitely generated projective R-modules (i.e. direct summands of K^n, $n = 0, 1, 2, \cdots$) and R-linear homomorphisms is a ribbon category (albeit not an interesting one from the viewpoint of application to knots):

- Braiding is given by flips (exchanges) $\beta_{V,W} = \tau_{V,W} : V \otimes W \to W \otimes V$.
- Twist is given by the identity endomorphism $\theta_V = \mathrm{id}_V$.
- The duals are the dual modules $V^* = \mathrm{Hom}_R(V, R)$, with obvious associated morphisms.

Example 5. Let G be a multiplicative abelian group, R a commutative ring with unit, $c : G \times G \to R^*$ a bilinear pairing (i.e. $c(gg', h) = c(g, h)\, c(g', h)$ and $c(g, hh') = c(g, h)\, c(g, h')$ for any $g, g', h, h' \in G$), $\varphi : G \to R^*$ a homomorphism such that $\varphi(g^2) = 1$ for all $g \in G$. Then, we can construct a ribbon category $\mathcal{V}(G, R, c, \varphi)$ as follows:

[1]As we will see later, this Proposition can be also seen as a corollary of a bigger theorem, Theorem 9.

- Objects are elements of G.
- Morphisms are given by

$$\mathrm{Hom}(g,h) = \begin{cases} R & \text{if } g = h, \\ \{0\} & \text{if } g \neq h. \end{cases}$$

- Composition of two morphisms $g \to h \to f$ is the product of the corresponding elements of R if $g = h = f$, and 0 otherwise.
- The tensor product of $g, h \in G$ is defined to be their product $gh \in G$.
- The tensor product $gg' \to hh'$ of two morphisms $g \to h$ and $g' \to h'$ is the product of the corresponding elements of R if $g = g'$ and $h = h'$ and 0 otherwise.
- Define the braiding $gh \to hg = gh$ to be $c(g,h) \in R$.
- Define the twist $g \to g$ to be $\varphi(g)\, c(g,g) \in R$.
- The dual object g^* is given by the inverse g^{-1}, and the associated morphisms are the endomorphisms of the unit of G represented by $1 \in R$.

Exercise 4. Show that $\mathcal{V}(G, R, c, \varphi)$ is indeed a ribbon category.

Example 6. For any *ribbon Hopf algebra* H, the category $\mathrm{Rep}(H)$ of finite-dimensional representations of H and H-linear homomorphisms is a ribbon category. We will study Hopf algebras later in this course.

5.1.1. *Trace and dimension.*

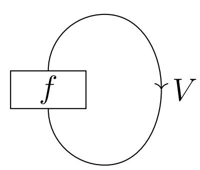
Definition 18. The *trace* of an endomorphism f of an object V of $\mathcal{V}$ is defined as

$$\mathrm{Tr}\,(f) := \overrightarrow{\cap}_V \circ (f \otimes \mathrm{id}_{V^*}) \circ \overleftarrow{\cup}_V \;\in\; \mathrm{End}(\mathbb{1}).$$

For any object V of $\mathcal{V}$, the *dimension* of V is defined to be

$$\dim\,(V) := \mathrm{Tr}\,(\mathrm{id}_V).$$

Graphically, the trace of $f : V \to V$ can be presented as



.

Proposition 10. *(1) For any morphisms $f : V \to W$, $g : W \to V$, we have*

$$\mathrm{Tr}\,(fg) = \mathrm{Tr}\,(gf).$$

(2) For any endomorphisms f, g of objects of $\mathcal{V}$, we have

$$\mathrm{Tr}\,(f \otimes g) = \mathrm{Tr}\,(f)\,\mathrm{Tr}\,(g).$$

(3) For any morhpism $k : \mathbb{1} \to \mathbb{1}$, we have

$$\mathrm{Tr}\,(k) = k.$$

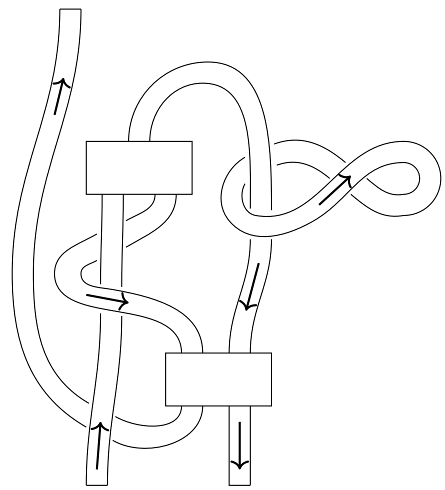
5.2. **Invariants of colored ribbon graphs.**

5.2.1. *Category of colored ribbon graphs.* Let $\mathcal{V}$ be a ribbon category.

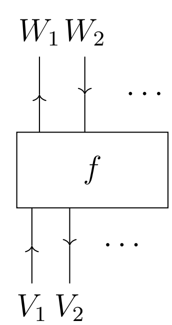
Definition 19. A *ribbon graph* is a compact oriented surface in $\mathbb{R}^3$ decomposed into bands, annuli, and coupons. Bands and annuli are oriented, and coupons have bottom and top bases. The ends of bands must lie on the bases of coupons.

We will also consider ribbon graphs in $\mathbb{R}^2 \times I$, in which case we allow the ends of the bands to end on either $\mathbb{R}^2 \times \{0\}$ or $\mathbb{R}^2 \times \{1\}$ as well.

See the figure below for an example of a ribbon graph in $\mathbb{R}^2 \times I$:



Definition 20. A $\mathcal{V}$*-coloring* of a ribbon graph is an assignment of an object of $\mathcal{V}$ to each band and annulus, and a morphism $f : V_1^{\epsilon_1} \otimes \cdots \otimes V_m^{\epsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n}$ to each coupon with m (resp. n) bands on the bottom (resp. top) with orientation $\epsilon_1, \cdots, \epsilon_m$ (resp. $\nu_1, \cdots, \nu_n$) colored by $V_1, \cdots, V_m$ (resp. $W_1, \cdots, W_n$). Here, $\epsilon = +1$ (resp. -1) corresponds to the upward (resp. downward) direction.



Definition 21. The category of colored ribbon graphs over $\mathcal{V}$, $\mathrm{Rib}_{\mathcal{V}}$, is defined as follows:

(1) The objects of $\mathrm{Rib}_{\mathcal{V}}$ are finite sequence
$$((V_1, \epsilon_1), \cdots, (V_m, \epsilon_m)), \quad m \in \mathbb{N},$$
where $V_1, \cdots, V_m$ are objects of $\mathcal{V}$ and $\epsilon_1, \cdots, \epsilon_m \in \{\pm 1\}$.

(2) A morphism $\eta \to \eta'$ in $\mathrm{Rib}_{\mathcal{V}}$ is an isotopy type of a colored ribbon graph in $\mathbb{R}^2 \times I$ such that η (resp. η') is the sequence of colors and directions of those bands which hit the bottom (resp. top) boundary intervals.

6. Lecture 6 (Thu Feb 8, 2024)

6.1. **Invariants of colored ribbon graphs (cont.)**

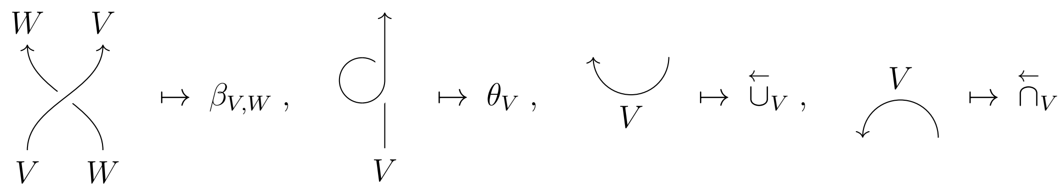
6.1.1. *Reshetikhin-Turaev functor.* The main theorem for our discussion of ribbon categories is the existence of the following functor, sometimes called the *Reshetikhin-Turaev functor*:

Theorem 9 ([Tur94, Thm 2.5])**.** *Let $\mathcal{V}$ be a strict ribbon category with braiding β, twist θ, and compatible duality $(*, b, d)$. Then there exists a unique monoidal functor*

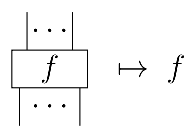
$$F = F_{\mathcal{V}} : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$$

satisfying the following conditions:

1. *F maps any object $(V, +1)$ to V and any object $(V, -1)$ to V^*.*
2. *F maps the string diagrams of braiding, twist, cups and caps (thought of as a colored ribbon graph) to the corresponding morphisms in $\mathcal{V}$:*



3. *F maps the elementary colored ribbon graph with coupon colored by f to f:*

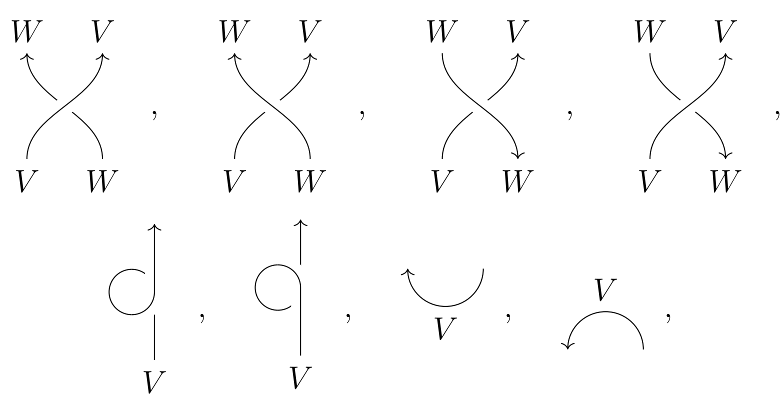


As a result, given a colored ribbon graph L, $F(L)$ is an isotopy invariant.

Remark 9. This is a far-reaching generalization of the Jones polynomial of links. In particular, when $\mathcal{V} = \mathrm{Rep}\ U_q(\mathfrak{sl}_2)^{\mathrm{fin}} \cong \mathrm{TLJ}(A)$ is the category of finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ and L is a link colored by the fundamental representation, then $F(L)$ is the Jones polynomial.

Proof of Theorem 9. The first step is the following lemmas which presents the category of colored ribbon graphs in terms of generators and relations:

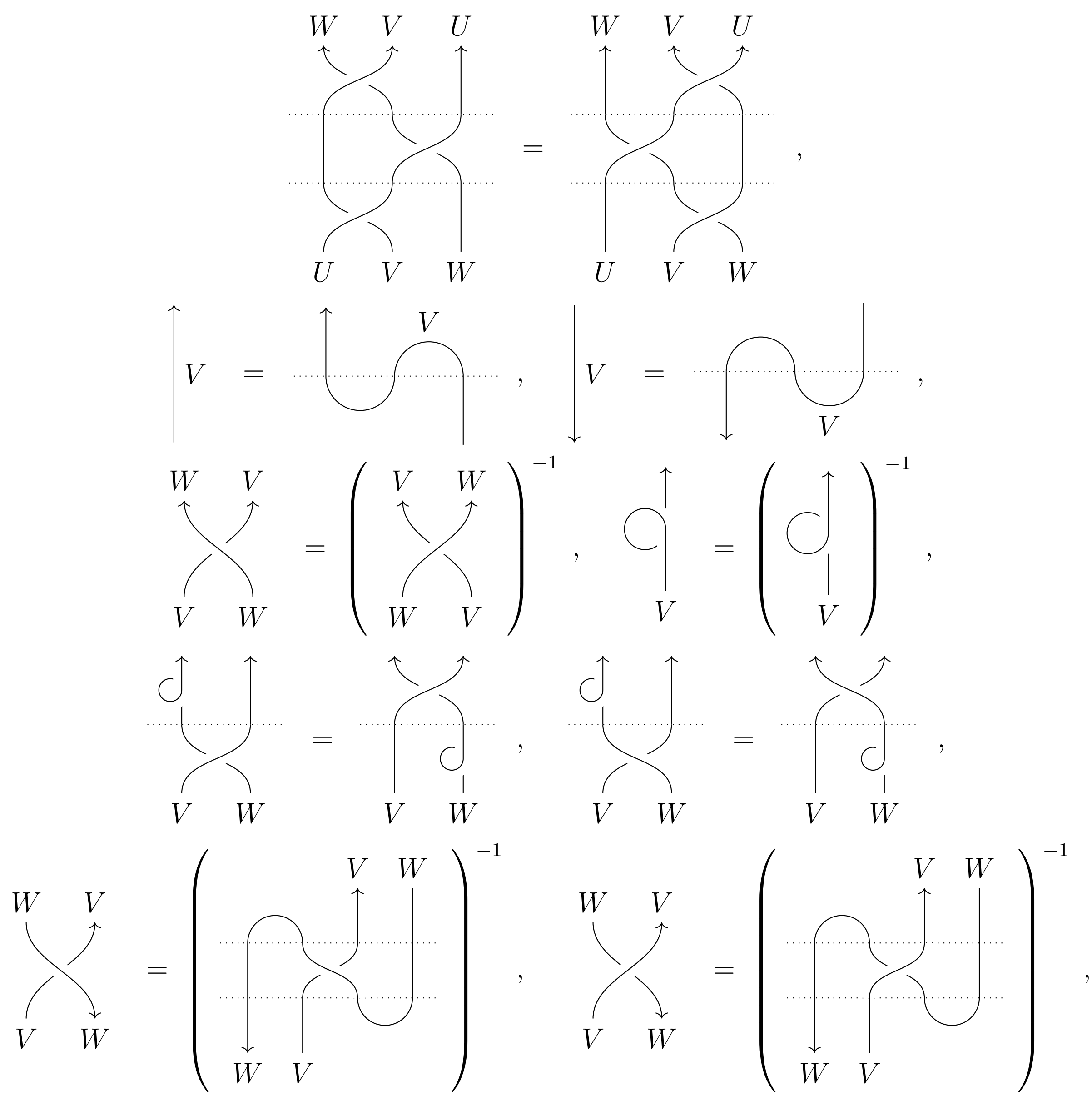
Lemma 1. *The category* $\mathrm{Rib}_{\mathcal{V}}$ *is generated by the colored ribbon tangles*

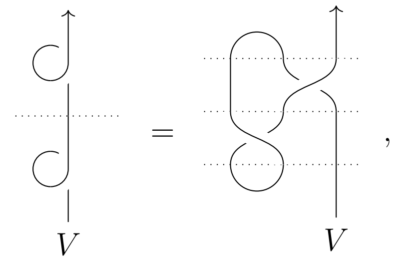


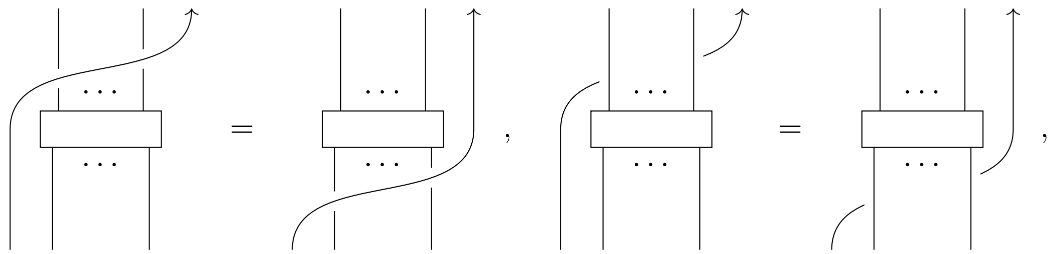
where V, W run over objects of $\mathcal{V}$, and all elementary colored ribbon graphs.

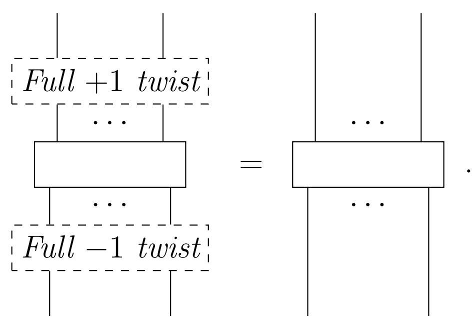
Proof. By choosing a generic diagram, we can decompose any colored ribbon graph into crossings, twists, cups and caps, and elementary colored ribbon graphs. Furthermore, by rotating crossings in wrong orientation, we can make all the crossings to be of the types given above. Finally, cups and caps in wrong orientation can be expressed in terms of the generators, as we've seen previously. □

Lemma 2. *The following relations form a complete set of relations between the generators of* $\mathrm{Rib}_{\mathcal{V}}$*:*



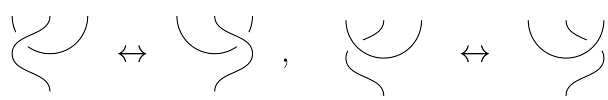






Proof sketch. Any two isotopic generic diagrams of colored ribbon tangles may be obtained from each other by a finite sequence of the following transformations:

(I) An isotopy in the class of generic diagrams.
(II) An isotopy interchanging the order of two singular points with respect to the height function.
(III) Birth or annihilation of a pair of local extrema.
(IV) Isotopies shown below:



Transformations of type (I) does not change the word at all. One can check that the transformation of words under transformations of types (II), (III) and (IV) can be derived from the relations given above.

In case of colored ribbon graphs (i.e. with coupons), any isotopy is a composition of isotopies of the following two kinds:

(i) Isotopies keeping the bases of all coupons horizontal.
(ii) $\pm 2\pi$ rotation of a coupon.

In fact, isotopies of type (ii) can be presented as compositions of isotopies of type (i). Furthermore, isotopies of type (i) can be presented as composition sof the following transformations:

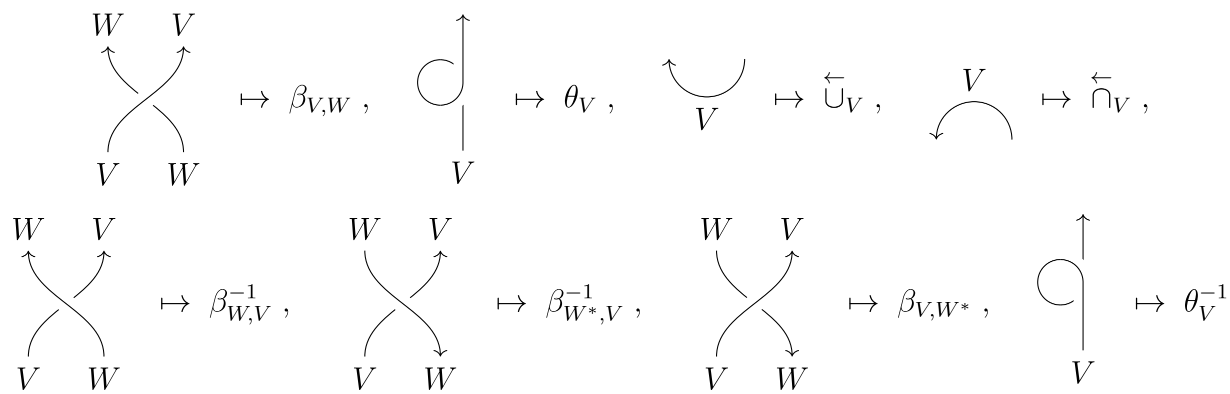
(iii) Ambient isotopies in $\mathbb{R} \otimes I$ keeping the bases of coupons horizontal.
(iv) The Reidemeister moves R1', R2 and R3 away from coupons.
(v) Isotopies which push a strand of the diagram over or under a coupon.

Again, one can check that the transformation of words under transformations of types (iii), (iv) and (v) can be derived from the relations given above. ☐

Now that we have a description of $\mathrm{Rib}_{\mathcal{V}}$ in terms of generators and relations, we need to check that the values of F on those generators respect the relations.

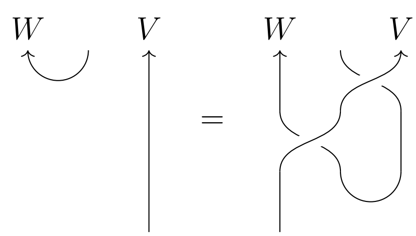
Uniqueness of F**:** From the conditions of the theorem, the value of F on any object $((V_1, \epsilon_1), \cdots, (V_m, \epsilon_m))$ of $\mathrm{Rib}_{\mathcal{V}}$ must be the object $V_1^{\epsilon_1} \otimes \cdots \otimes V_m^{\epsilon_m}$ of $\mathcal{V}$, and the value of the generators of $\mathrm{Rib}_{\mathcal{V}}$ are uniquely determined. This implies uniqueness of the functor.

Existence of F**:** Existence can be shown by checking that the assignments

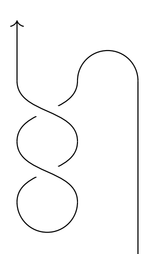


satisfy all the relations between the generators:

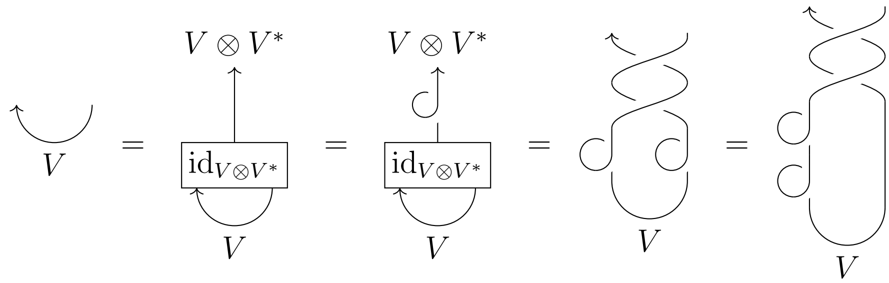
- The first one is the Yang-Baxter equation, which we verified earlier.
- The second and third relations follow from the definition of duality, and the fourth and fifth relations are immediate from our definition of the value of F on those generators.
- The sixth and seventh relations follow from naturality of braiding.
- The eighth and ninth relations can be reduced to showing the relation



  and similar diagrams, but this just follows from the naturality of braiding.
- Using the previous relation, the tenth relation can be reduced to showing the relation

$\theta_V^2 =$  .

This follows from the following identity which follows from naturality of the twist and its compatibility with duality:



- The last three relations involve a coupon. For the first two, we can first re-orient all the downward-oriented strands upward and replace the colors with their dual objects. This does not change their value under F. Once all the strands are oriented upward, the relations just follow from naturality of the braiding.
- For the very last relation, we may again re-orient all the strands so that they are all oriented upward. Thanks to the naturality of twist, it suffices to show that the tangle corresponding to a full positive twist, with strands colored by $V_1, \cdots, V_m$, evaluate to $\theta_{V_1 \otimes \cdots, \otimes V_m}$. This can be seen easily from the defining property of the twist (and induction on the number of strands).

□

7. Lecture 7 (Tue Feb 13, 2024)

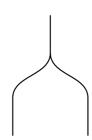
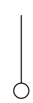
7.1. **Hopf algebras.** We follow [Kas23, Ch. 1-5] and [KRT97, Ch. 2-4].

7.1.1. *Algebras and coalgebras.* Let's start by reviewing basic definitions of algebras and coalgebras.

Definition 22. An *algebra* over a field $\Bbbk$, or a $\Bbbk$-*algebra*, is a triple (A, μ, η) consisting of a $\Bbbk$-vector space A, a linear map $\mu : A \otimes A \to A$ called *product*, and a linear map $\eta : \Bbbk \to A$ called *unit* such that

(1) $\mu(\mu \otimes \mathrm{id}_A) = \mu(\mathrm{id}_A \otimes \mu)$,
(2) $\mu(\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = \mu(\mathrm{id}_A \otimes \eta)$.

In terms of string diagrams, we will denote the product and the unit as

μ =:  , η =: 

Definition 23. The *opposite product* of an algebra $A := (A, \mu, \eta)$ is the linear map

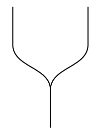
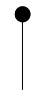
$$\mu^{\mathrm{op}} := \mu\, \sigma_{A,A} : A \otimes A \to A$$
$$x \otimes y \mapsto \mu(y \otimes x).$$

Proposition 11. *If $A := (A, \mu, \eta)$ is an algebra, then its* opposite algebra $A^{\mathrm{op}} := (A, \mu^{\mathrm{op}}, \eta)$ *is also an algebra.*

Definition 24. A *coalgebra* over a field $\Bbbk$, or a $\Bbbk$-coalgebra, is a triple (C, Δ, ϵ) consisting of a $\Bbbk$-vector space C, a linear map $\Delta : C \to C \otimes C$ called *coproduct*, and a linear map $\epsilon : C \to \Bbbk$ called *counit* such that

(1) $(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta$,
(2) $(\epsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C = (\mathrm{id}_C \otimes \epsilon)\Delta$.

In terms of string diagrams, we will denote the coproduct and the counit as

$\Delta =:$  , $\epsilon =:$ 

Notation 1 (Sweedler's sigma notation for coalgebras)**.** Sweedler's *sigma notation* allows us to write formally the coproduct of an element of a coalgebra in the form

$$\Delta x = \sum_{(x)} x_{(1)} \otimes x_{(2)},$$

where the meaning of the sum is that it is a finite sum of the form $\Delta x = \sum_{i=1}^{n} a_i \otimes b_i$.

Likewise, iterated coproducts $\Delta^{(m)} := (\Delta^{(m-1)} \otimes \mathrm{id}_C)\Delta$ (with $\Delta^{(0)} = \epsilon$, $\Delta^{(1)} = \mathrm{id}_C$) can be written formally as

$$\Delta^{(m)} x = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(m)}.$$

Definition 25. The *opposite coproduct* of a coalgebra $C := (C, \Delta, \epsilon)$ is the linear map

$$\begin{aligned} \Delta^{\mathrm{op}} := \sigma_{C,C}\Delta : C &\to C \otimes C \\ x &\mapsto \sum_{(x)} x_{(2)} \otimes x_{(1)}. \end{aligned}$$

Proposition 12. *If $C := (C, \Delta, \epsilon)$ is a coalgebra, then its* opposite coalgebra $C^{\mathrm{cop}} := (C, \Delta^{\mathrm{op}}, \epsilon)$ *is also a coalgebra.*

Theorem 10 (The Fundamental theorem of coalgebras)**.** *Let $C = (C, \Delta, \epsilon)$ be a coalgebra and $x \in C$. Then, there exists a finite-dimensional sub-coalgebra $X \subset C$ containing x.*

Remark 10. There's no counterpart of this theorem for algebras. For instance, $x \in \mathbb{C}[x]$ is a simple counterexample.

Proof of Theorem 10. Let $\{\alpha_i\}_{i\in I}$ and $\{\beta_j\}_{j\in J}$ be two finite sets of linearly independent elements of C such that

$$\Delta^{(3)}(x) = \sum_{\substack{i\in I \\ j\in J}} \alpha_i \otimes x_{i,j} \otimes \beta_j$$

for some $x_{i,j} \in C$. Let $X \subset C$ be the vector subspace spanned by $\{x_{i,j}\}_{(i,j)\in I\times J}$. We will show that X is a sub-coalgebra of C, and for that we need to show $\Delta(X) \subset X \otimes X$.

Firstly, note that

$$\sum_{\substack{i\in I \\ j\in J}} \alpha_i \otimes (\Delta x_{i,j}) \otimes \beta_j \;=\; \Delta^{(4)} x \;=\; \sum_{\substack{k\in I \\ j\in J}} (\Delta \alpha_k) \otimes x_{k,j} \otimes \beta_j.$$

By linear independence of β_j's, we have

$$\sum_{i\in I} \alpha_i \otimes \Delta x_{i,j} \;=\; \sum_{k\in I} \Delta \alpha_k \otimes x_{k,j},$$

and linear independence of α_i's imply that

$$\Delta\alpha_k = \sum_{i\in I} \alpha_i \otimes \alpha_{i,k} \ , \quad \Delta x_{i,j} = \sum_{k\in I} \alpha_{i,k} \otimes x_{k,j}$$

for some $\{\alpha_{i,k}\}_{i,k\in I} \subset C$. Therefore, $\Delta(X) \subset C \otimes X$.

Likewise,

$$\sum_{\substack{i\in I\\ j\in J}} \alpha_i \otimes (\Delta x_{i,j}) \otimes \beta_j \ = \ \Delta^{(4)}x \ = \ \sum_{\substack{i\in I\\ l\in J}} \alpha_i \otimes x_{i,l} \otimes (\Delta\beta_l),$$

and by a similar argument, we can deduce that $\Delta(X) \subset X \otimes C$ as well.

Therefore, $\Delta(X) \subset C \otimes X \cap X \otimes C = X \otimes X$. □

7.1.2. *Convolution algebras.*

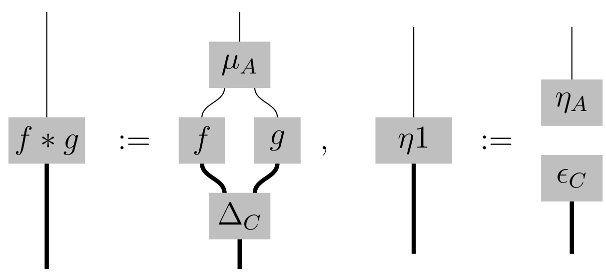
Definition 26. Let A be an algebra and C a coalgebra. The *convolution algebra* $\mathrm{L}(C, A)$ is the vector space of linear maps from C to A, with the product $\mu : \mathrm{L}(C, A)\otimes\mathrm{L}(C, A) \to \mathrm{L}(C, A)$ defined by

$$\mu(f \otimes g) =: f * g := \mu_A(f \otimes g)\Delta_C,$$

and the unit given by $\eta 1 := \eta_A\epsilon_C$.

Proposition 13. *The convolution algebra* $\mathrm{L}(C, A)$ *is an algebra.*

Proof. Diagrammatically, the convolution product and the unit are given by



It is straightforward to check that the associativity and unitality of the convolution algebra follows from those of the algebra A and coassociativity and counitality of the coalgebra C. □

Corollary 1. *The dual space* $C^* = \mathrm{L}(C, \Bbbk)$ *of a coalgebra* C *is an algebra with the convolution product*

$$\mu_{C^*} = \Delta^*|_{C^*\otimes C^*}.$$

As we will see, a lot of concepts that we will encounter can be nicely phrased in terms of a convolution algebra.

7.1.3. *Restricted (or finite) dual of an algebra.* White the dual space C^* of a coalgebra C is always an algebra (Corollary 1), it is not always true that the dual space A^* of an algebra A is a coalgebra, when A is infinite dimensional. This is because the image of

$$\mu^* : A^* \to (A \otimes A)^* \supset A^* \otimes A^*$$

does not always lie in $A^* \otimes A^*$. This motivates the following definition:

Definition 27. The *restricted (or finite) dual* A^o of an algebra A is the vector subspace of A^* given by

$$A^o := (\mu^*)^{-1}(A^* \otimes A^*).$$

It turns out that the restricted dual is nicely characterized by matrix coefficients of finie dimensional representations of A.

Theorem 11. *The restricted dual A^o of any algebra A is the linear span of the matrix coefficients of all finite dimensional representations of A.*

Proof. Suppose that $\lambda : A \to \mathrm{End}(V)$ is an n-dimensional (left) representation of A so that, with respect to some basis $\{v_i\}_{1\leq i\leq n}$ of V,

$$(\lambda x)v_i = \sum_{1\leq j\leq n} v_j\langle \lambda_{j,i}, x\rangle$$

for any $x \in A$. Then,

$$\sum_{1\leq j\leq n} v_j\langle \mu^*\lambda_{j,i}, x\otimes y\rangle = \sum_{1\leq j\leq n} \langle \lambda_{j,i}, xy\rangle = (\lambda(xy))v_i$$
$$= (\lambda x)(\lambda y)v_i = \sum_{1\leq k\leq n} (\lambda x)v_k\langle \lambda_{k,i}, y\rangle = \sum_{1\leq j,k\leq n} v_j\langle \lambda_{j,k}, x\rangle\langle \lambda_{k,i}, y\rangle = \sum_{1\leq j,k\leq n} v_j\langle \lambda_{j,k}\otimes\lambda_{k,i}, x\otimes y\rangle.$$

In other words,

$$\mu^*\lambda_{j,i} = \sum_{1\leq k\leq n} \lambda_{j,k}\otimes\lambda_{k,i},$$

and the matrix coefficients $\lambda_{j,i} \in A^*$ lie in A^o.

Now, it suffices to show that, for any element $f \in A^o$, there exists a finite dimensional (left) A-module V_f such that f is a linear combination of the matrix coefficients of this representation, with respect to some basis. For this, consider the dual space A^* as a left A-module, with the left A-action given by the dual right multiplications $R^*_x \in \mathrm{End}(A^*)$. Let

$$V_f := R^*_A f \ \subset \ A^*$$

be the vector subspace given by the orbit of f with respect to this action of A on A^*. That is, we have an algebra morphism

$$\lambda : A \to \mathrm{End}(V_f).$$

The condition $f \in A^o$ implies that

$$\mu^* f = \sum_{1\leq i\leq n} g_i\otimes h_i$$

for some n and $g_i, h_i \in A^*$. Since

$$\langle R^*_x f, y\rangle = \langle f, yx\rangle = \langle \mu^* f, y\otimes x\rangle = \sum_{1\leq i\leq n} \langle g_i, y\rangle\langle h_i, x\rangle = \left\langle \sum_{1\leq i\leq n} g_i\langle h_i, x\rangle, y\right\rangle,$$

we have

$$R^*_x f = \sum_{1\leq i\leq n} g_i\langle h_i, x\rangle,$$

and in particular, V_f is in the linear span of $\{g_i\}_{1\leq i\leq n}$; it is finite dimensional.

Let $\{v_i\}_{1\leq i\leq m}$ be a linear basis of $V_f \subset A^*$. Then, for any $x \in A$,

$$R^*_x f = \sum_{1\leq i\leq m} v_i\langle w_i, x\rangle$$

for some $w_i \in A^*$. Note,

$$f = R_1^* f = \sum_{1\le i\le m} v_i \langle w_i, 1\rangle, \tag{1}$$

and also

$$\langle f, x\rangle = \langle R_x^* f, 1\rangle = \sum_{1\le i\le m} \langle v_i, 1\rangle \langle w_i, x\rangle = \Big\langle \sum_{1\le i\le m} \langle v_i, 1\rangle w_i, x \Big\rangle,$$

which implies

$$f = \sum_{1\le i\le m} \langle v_i, 1\rangle w_i. \tag{2}$$

Let $z_i \in A$ be some vectors dual to w_i in such a way that

$$R_{z_i}^* f = v_i.$$

Then we have

$$(\lambda x) v_i \;=\; (\lambda x) R_{z_i}^* f \;=\; R_{xz_i}^* f \;=\; \sum_{1\le j\le m} v_j \langle w_j, x z_i\rangle \;=\; \sum_{1\le j\le m} v_j \langle R_{z_i}^* w_j, x\rangle,$$

i.e.

$$\lambda_{i,j} = R_{z_j}^* w_i.$$

Finally, f is a linear combination of $\lambda_{i,j}$'s, because

$$\begin{aligned} f &\overset{(1)}{=} \sum_{1\le j\le m} v_j \langle w_j, 1\rangle = \sum_{1\le j\le m} R_{z_j}^* f \langle w_j, 1\rangle \\ &\overset{(2)}{=} \sum_{1\le i,j\le m} \langle v_i, 1\rangle \langle w_j, 1\rangle R_{z_j}^* w_i = \sum_{1\le i,j\le m} \langle v_i, 1\rangle \langle w_j, 1\rangle \lambda_{i,j}. \end{aligned}$$

□

Corollary 2. *For any algebra A,*

$$\mu^*(A^o) \subset A^o \otimes A^o.$$

Therefore, A^o is a coalgebra.

Remark 11. There is a natural equivalence

$$\mathrm{Hom}_{\mathrm{Alg}_{\Bbbk}}(A, C^*) \simeq \mathrm{Hom}_{\mathrm{Coalg}_{\Bbbk}}(C, A^o)$$

for all algebra A and coalgebra C.

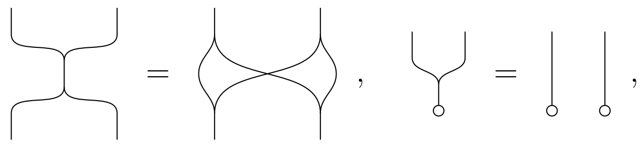
8. Lecture 8 (Thu Feb 15, 2024)

8.1. **Hopf algebras (cont.)**

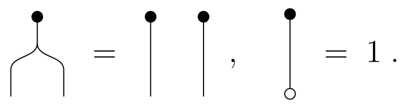
8.1.1. *Bialgebras and Hopf algebras.*

Definition 28. A *bialgebra* is a tuple $(B, \mu, \eta, \Delta, \epsilon)$, where (B, μ, η) is an algebra, (B, Δ, ϵ) is a coalgebra, and the linear maps Δ and ϵ are algebra morphisms (or, equivalently, μ and η are coalgebra morphisms).

In terms of string diagrams, that Δ is an algebra morphism means



and that ϵ is an algebra morhpism means

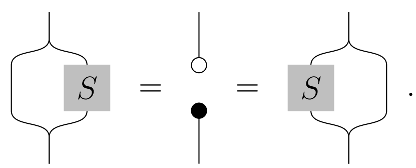


Example 7. For any monoid M, the monoid algebra $\Bbbk[M]$ is a bialgebra with coproduct $\Delta(x) = x \otimes x$ and counit $\epsilon(x) = 1$, for all $x \in M$.

Definition 29. A *Hopf algebra* (over a field $\Bbbk$) is a bialgebra H such that the identity map id_H is invertible in the convolution algebra $\mathrm{End}(H)$ of endomorphisms of H.

The convolution inverse of id_H is denoted by S and is called the *antipode* of H.

In terms of string diagrams,



Commutative and cocommutative Hopf algebras are closely related to groups and Lie algebras.

Example 8. For any group G, the group algebra $\Bbbk[G]$ is a cocommutative Hopf algebra with

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}$$

for any $g \in G$.

It's dual algebra $\Bbbk[G]^*$ is the commutative algebra $\Bbbk^G$ of $\Bbbk$-valued functions on G. If G is finite, $\Bbbk^G$ is a Hopf algebra with

$$\Delta(f)(g \otimes h) = f(gh), \quad \epsilon(f) = f(e), \quad S(f)(g) = f(g^{-1})$$

for any $f \in \Bbbk^G$.

Example 9. Let V be a vector space. Then, the tensor algebra $T(V)$ is a cocommutative Hopf algebra, with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x$$

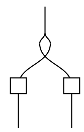
for any $x \in V$.

Example 10. Let $\mathfrak{g}$ be a Lie algebra. Then, the universal enveloping algebra $U(\mathfrak{g})$, which is a quotient of $T(\mathfrak{g})$ by relations $x \otimes y - y \otimes x = [x, y]$, is also a cocommutative Hopf algebra.

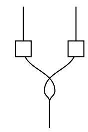
Below, we list some properties of a Hopf algebra.

Proposition 14. *In any Hopf algebra $H = (H, \mu, \eta, \Delta, \epsilon, S)$, the product μ (resp. the coproduct Δ) is an invertible element of the convolution algebra $L(H \otimes H, H)$ (resp. $L(H, H \otimes H)$) with the inverse*

$$\overline{\mu} := \mu^{\mathrm{op}}(S \otimes S) =$$



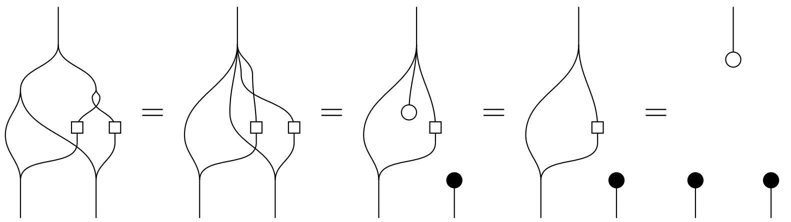
$$\left(\textit{resp. } \overline{\Delta} := (S \otimes S)\Delta^{\mathrm{op}} = \right.$$



$$\left. \right).$$

Proof. Here's a graphical proof that $\mu * \overline{\mu}$ is the convolution identity:

$$\mu * \overline{\mu} =$$



The rest of the proof is similar. ∎

Exercise 5. Show that if $H := (H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra, then

$$H^{\mathrm{op,cop}} := (H, \mu^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \epsilon, S)$$

is also a Hopf algebra.

Exercise 6. Show that if $H = (H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra with invertible antipode S, then

$$H^{\mathrm{cop}} := (H, \mu, \eta, \Delta^{\mathrm{op}}, \epsilon, S^{-1})$$

and

$$H^{\mathrm{op}} := (H, \mu^{\mathrm{op}}, \eta, \Delta, \epsilon, S^{-1})$$

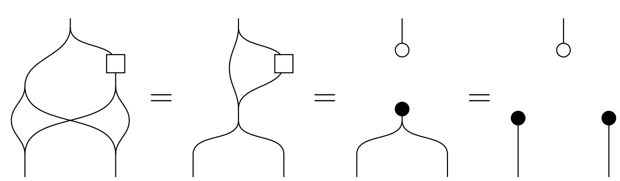
are also Hopf algebras.

Proposition 15. *In any Hopf algebra H, the antipode is a Hopf algebra morphism from H to $H^{\mathrm{op,cop}}$. That is,*

$$\begin{aligned} S\mu &= \mu^{\mathrm{op}}(S \otimes S), \\ \Delta S &= (S \otimes S)\Delta^{\mathrm{op}}, \\ \epsilon S &= \epsilon, \\ S\eta &= \eta. \end{aligned}$$

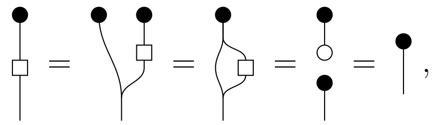
Proof. Note that the RHS of the first two identities are the convolution inverses of the product and coproduct that appeared in Proposition 14. By uniqueness of convolution inverse, it suffices to show that $S\mu$ and ΔS are also convolution inverses of μ and Δ, respectively. This can be checked easily using graphical calculus:

$$\mu * (S\mu) =$$



and likewise for other identities.

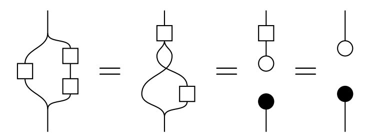
For the third and fourth identity,

$\epsilon S =$ 

and similarly for $S\eta = \eta$. □

Corollary 3. *If H is either commutative or cocommutative, then the antipode is involutory (i.e. $S^2 = \mathrm{id}_H$).*

Proof. Thanks to the uniqueness of convolution inverse, it suffices to check that S^2 is the convolution inverse of S. Using Proposition 15 and commutativity (or cocommutativity), we have

$S * S^2 =$ 

and likewise, $S^2 * S = \eta\epsilon$. Therefore, $S^2 = \mathrm{id}_H$ whenever H is either commutative or cocommutative. □

The bialgebra and Hopf algebra structures are exactly what is necessary for for an algebra for its representations to have tensor products and duals:

Proposition 16. *Let $B = (B, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. Then, the category $B-\mathrm{Mod}$ of left B-modules form a monoidal category.*

Proof. The coproduct Δ equips the tensor product $U \otimes V$ of two B-module a B-module structure by

$$b(u \otimes v) := \Delta(b)(u \otimes v) = \sum_{(b)} b_{(1)}u \otimes b_{(2)}v,$$

for any $b \in B$, $u \in U$ and $v \in V$. Moreover, the counit ϵ equips the ground field $\Bbbk$ with a trivial B-module structure by

$$b\, z := \epsilon(b)\, z$$

for any $b \in B$ and $z \in \Bbbk$.

For three B-modules U, V, W, we have the following canonical isomorphisms in $B-\mathrm{Mod}$:

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$$
$$(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$$

and

$$\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$$
$$1 \otimes v \mapsto v \mapsto v \otimes 1.$$

One can easily check that that these satisfy the axioms of a monoidal category. □

Proposition 17. *Let H be a Hopf algebra. Then, the category $H-\mathrm{Mod}^{\mathrm{fin}}$ of finite dimensional left H-modules form a rigid monoidal category (i.e. every object has duals).*

Proof. The antipode S equips the dual space V^* of any left H-module V an H-module structure by

$$\langle x\, f, v\rangle := \langle f, S(x)v\rangle$$

for any $x \in H$, $f \in V^*$, and $v \in V$. This is indeed an H-module structure, because

$$\langle (xy)\, f, v\rangle = \langle f, S(xy)v\rangle = \langle f, S(y)S(x)v\rangle = \langle x(y\, f), v\rangle.$$

It is easy to check that the dual representation V^* is indeed a dual object of V, with the duality morphisms

$$\begin{aligned} \mathrm{ev}_V : V \otimes V^* &\to \Bbbk \\ v \otimes f &\mapsto f(v) \end{aligned}$$

and

$$\begin{aligned} \mathrm{coev}_V : \Bbbk &\to V^* \otimes V \\ z &\mapsto z \sum_{i\in I} e^i \otimes e_i, \end{aligned}$$

where $\{e_i\}_{i\in I}$ is a basis of V and $\{e^i\}_{i\in I}$ is a dual basis of V^*. □

When we are dealing with infinite dimensional Hopf algebras, the appropriate notion of dual is the restricted dual as we did for algebras:

Definition 30. The *restricted dual* H^o of a Hopf algebra H is defined as the restricted dual of the underlying algebra.

Proposition 18. *For any Hopf algebra $H = (H, \mu, \eta, \Delta, \epsilon, S)$, the restricted dual H^o is a Hopf algebra with respect to the dual structural maps:*

$$\mu_{H^o} = \Delta^*|_{H^o\otimes H^o}, \quad \eta_{H^o} = \epsilon^* : 1 \mapsto \epsilon, \quad \Delta_{H^o} = \mu^*|_{H^o}, \quad \epsilon_{H^o} = \eta^o = \eta^*|_{H^o}, \quad S_{H^o} = S^o = S^*|_{H^o}.$$

Example 11. The restricted dual of the Hopf algebra $\mathbb{C}[x]$ where x is a primitive element with coproduct $\Delta(x) = 1 \otimes x + x \otimes 1$ is given by

$$\mathbb{C}[x]^o \cong \mathbb{C}[\mathbb{C}] \otimes \mathbb{C}[t].$$

Here,

$$\rho_z := \chi_z \otimes 1, \quad \partial := 1 \otimes t$$

are given by

$$\langle \partial, x^n\rangle = \delta_{1,n}, \quad \langle \rho_z, x^n\rangle = z^n.$$

9. Lecture 9 (Tue Feb 20, 2024)

9.1. **Hopf algebras (cont.)**

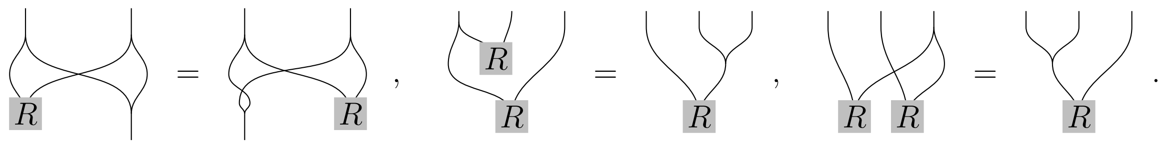
9.1.1. *Braided bialgebras.*

Definition 31. A *universal R-matrix* in a bialgebra $B = (B, \mu, \Delta, \eta, \epsilon)$ is an invertible element $R \in B \otimes B$ such that

(1) $R * \Delta = \Delta^{\mathrm{op}} * R$,
(2) $R_{13} * R_{12} = (1 \otimes \Delta)R$,
(3) $R_{13} * R_{23} = (\Delta \otimes 1)R$.

A bialgebra provided with a universal R-matrix is called *braided* (or *quasitriangular*).

In terms of string diagrams,



Observe that any cocommutataive bialgebra is braided with universal R-matrix $R = 1 \otimes 1$.

Theorem 12. *For a bialgebra B, the monoidal category $B-\mathrm{Mod}$ is braided if and only if there exists a universal R-matrix.*

Proof. Let B be a braided bialgebra with universal R-matrix R. Then for all pairs (V, W) of left B-modules, we define a natural isomorphism $\beta_{V,W} : V \otimes W \to W \otimes V$ of B-modules by

$$\begin{aligned}\beta_{V,W} : V \otimes W &\to W \otimes V\\ v \otimes w &\mapsto \sigma_{V,W}(R(v \otimes w)).\end{aligned}$$

We claim that the family $\{\beta_{V,W}\}$ is a braiding in $B-\mathrm{Mod}$. Firstly, it is indeed a morphism in $B-\mathrm{Mod}$ (i.e. it is B-linear) because, for any $x \in B$,

$$\begin{aligned}\beta_{V,W}(x(v \otimes w)) &= \sigma_{V,W}(R\Delta(x)(v \otimes w))\\ &= \sigma_{V,W}(\Delta^{\mathrm{op}}(x)R(v \otimes w))\\ &= \Delta(x)\sigma_{V,W}(R(v \otimes w))\\ &= x\,\beta_{V,W}(v \otimes w).\end{aligned}$$

Next, $\beta_{V,W}$ is an isomorphism, with the inverse given by

$$\beta^{-1}_{V,W}(w \otimes v) = R^{-1}(v \otimes w).$$

Finally, for any triple of objects U, V, W,

$$\begin{aligned}\beta_{U\otimes V,W}(u \otimes v \otimes w) &= \sigma_{U\otimes V,W}(R((u \otimes v) \otimes w))\\ &= \sigma_{U\otimes V,W}((\Delta \otimes 1)R(u \otimes v \otimes w))\\ &= \sigma_{U,W}(\sigma_{V,W}(R_{13}R_{23}(u \otimes v \otimes w)))\\ &= \sigma_{U,W}(R_{12}(\sigma_{V,W}(R_{23}(u \otimes v \otimes w))))\\ &= (\beta_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \beta_{V,W})(u \otimes v \otimes w)\end{aligned}$$

and likewise,

$$\beta_{U,V\otimes W} = (\mathrm{id}_V \otimes \beta_{U,W})(\beta_{U,V} \otimes \mathrm{id}_W).$$

Therefore, $\{\beta_{V,W}\}$ is a braiding in $B-\mathrm{Mod}$.

Conversely, suppose there exists a braiding β in $B-\mathrm{Mod}$. Define an element $R \in B \otimes B$ by

$$R := \sigma_{B,B}(\beta_{B,B}(1 \otimes 1)).$$

Let's show that R is a universal R-matrix in B. For any elements v, w of B-modules V, W, let $\overline{v} : B \to V$ and $\overline{w} : B \to W$ be the B-linear maps uniquely determined by $\overline{v}(1) = v$ and $\overline{w}(1) = w$. Then, by naturality of braiding,

$$\begin{aligned}\beta_{V,W}(v \otimes w) &= \beta_{V,W}(\overline{v} \otimes \overline{w})(1 \otimes 1) = (\overline{w} \otimes \overline{v})\beta_{B,B}(1 \otimes 1)\\ &= \sigma_{V,W}(\overline{v} \otimes \overline{w})R = \sigma_{V,W}(R(v \otimes w))\end{aligned}$$

Then, by B-linearity of the braiding $\beta_{B,B}$, we get

$$\Delta(x)\sigma_{B,B}(R) = \sigma_{B,B}(R\Delta(x))$$

for any $x \in B$. This verifies property (1) of the universal R-matrix.

The remaining properties (2 and 3) of the universal R-matrix follow from the properties

$$\begin{aligned}\beta_{U,V\otimes W} &= (\mathrm{id}_V \otimes \beta_{U,W})(\beta_{U,V} \otimes \mathrm{id}_W),\\ \beta_{U\otimes V,W} &= (\beta_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes \beta_{V,W})\end{aligned}$$

of the braiding. □

Exercise 7. Show that the braiding $\beta_{V,W} = \sigma_{V,W}R$ satisfies $\beta_{W,V}\beta_{V,W} = \mathrm{id}_{V\otimes W}$ for all modules V, W if and only if $R^{-1} = \sigma_{A,A}(R)$.

Here are a few properties of the universal R-matrix:

Proposition 19. *Let B be a braided bialgebra with universal R-matrix R. Then,*

(1) it satisies the Yang-Baxter relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

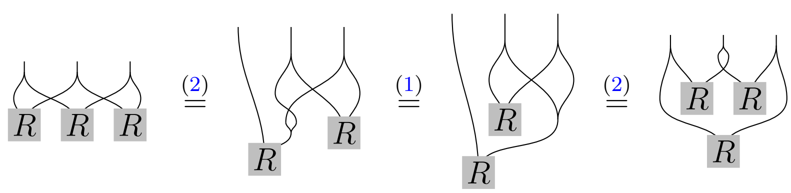
(2)

$$(\epsilon \otimes \mathrm{id}_B)R = 1 = (\mathrm{id}_B \otimes \epsilon)R,$$

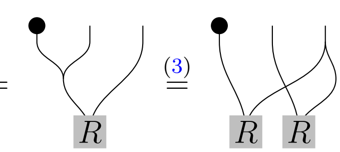
(3) and if B has an antipode S (so that it is a Hopf algebra), then

$$R^{-1} = (S \otimes \mathrm{id}_B)R.$$

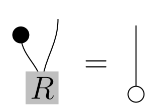
Proof. For the first relation,

$R_{12}R_{13}R_{23} =$  $= R_{23}R_{13}R_{12}.$

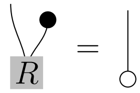
For the second relation, first note that

$R =$  .

By invertibility of R, it follows that

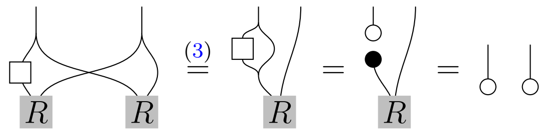
 .

The other equality



can be shown in a similar way.

The third relation is true because

$((S \otimes \mathrm{id}_B)R)R =$ 

∎

Proposition 20. *Let $(B, \mu, \eta, \Delta, \epsilon, S, R)$ be a braided Hopf algebra with an invertible antipode. Consider the element*

$$u := \mu((S \otimes \mathrm{id}_B)\sigma_{B,B}(R)) = \sum_i S(t_i)s_i \ \in B,$$

where $R = \sum_i s_i \otimes t_i$. Then

(1) u is invertible with inverse

$$u^{-1} = \mu((S^{-1} \otimes S)\sigma_{B,B}(R)) = \sum_i S^{-1}(t_i)S(s_i),$$

and satisfies

$$\Delta(u) = (R_{21}R)^{-1}(u \otimes u),$$

(2) $S^2(x) = uxu^{-1}$ for all $x \in B$ (i.e. S^2 is an inner automorphism),

(3) $uS(u) = S(u)u \ \in B$ is central and satisfies

$$S(uS(u)) = uS(u), \quad \epsilon(uS(u)) = 1, \quad \Delta(uS(u)) = (R_{21}R)^{-2}(uS(u) \otimes uS(u)).$$

Definition 32. A *ribbon Hopf algebra* is a braided Hopf algebra equipped with an invertible central element ν (called a *ribbon element*) satisfying

$$\nu^2 = uS(u), \quad S(\nu) = \nu, \quad \epsilon(\nu) = 1, \quad \Delta(\nu) = (R_{21}R)^{-1}(\nu \otimes \nu).$$

Theorem 13. *Let H be a ribbon Hopf algebra. Then the category $H{-}\mathrm{Mod}^{\mathrm{fin}}$ of finite dimensional left H-modules form a ribbon category.*

Proof. We already know from Proposition 17 that $H{-}\mathrm{Mod}^{\mathrm{fin}}$ is a rigid monoidal category, so we just need to define the twist and show that it is compatible with the duality.

For each H-module V, define

$$\begin{aligned} \theta_V : V &\to V \\ v &\mapsto \nu^{-1} v. \end{aligned}$$

This is an H-module isomorphism because ν^{-1} is central and invertible. Moreover, this is a twist, because

$$\begin{aligned} \theta_{V \otimes W}(v \otimes w) &= \Delta(\nu^{-1})(v \otimes w) \\ &= R_{21}R(\nu^{-1} \otimes \nu^{-1})(v \otimes w) \\ &= \sigma_{W,V} R \sigma_{V,W} R(\nu^{-1} \otimes \nu^{-1})(v \otimes w) \\ &= \beta_{W,V}\beta_{V,W}(\theta_V \otimes \theta_W)(v \otimes w). \end{aligned}$$

Let's check the compatibility of this twist with duality. We need to check

$$(\theta_V \otimes \mathrm{id}_{V^*})\,\mathrm{coev}_V = (\mathrm{id}_V \otimes \theta_{V^*})\,\mathrm{coev}_V.$$

Fix a basis $\{e_i\}_{i\in I}$ of V and a dual basis $\{e^i\}_{i\in I}$ of V^*. Let's write

$$\nu e_i = \sum_{j\in I} \nu_i^j e_j$$

for some matrix coefficients $\nu_i^j \in \Bbbk$. Then

$$\langle \nu e^j, e_i\rangle = \langle e^j, S(\nu) e_i\rangle = \langle e^j, \nu e_i\rangle = \nu_i^j,$$

which means

$$\nu e^j = \sum_{i\in I} \nu_i^j e^i.$$

Then, for any $z \in \Bbbk$,

$$\begin{aligned}(\theta_V \otimes \mathrm{id}_{V^*})\,\mathrm{coev}_V(z) &= z(\theta_V \otimes \mathrm{id}_{V^*}) \sum_{i\in I} e_i \otimes e^i \\ &= z \sum_{i\in I} \nu e_i \otimes e^i \\ &= z \sum_{i,j\in I} \nu_i^j e_j \otimes e^i \\ &= z \sum_{j\in I} e_j \otimes \nu e^j \\ &= (\mathrm{id}_V \otimes \theta_{V^*})\,\mathrm{coev}_V(z).\end{aligned}$$

Therefore, the twist is compatible with duality. □

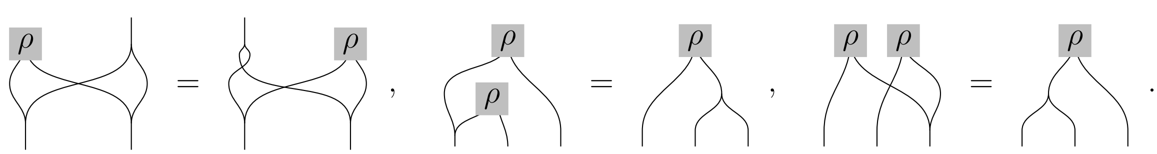
9.1.2. *Cobraided bialgebras.* Sometimes, it will be more convenient to work with the following dual notion:

Definition 33. A *dual universal R-matrix* in a bialgebra $B = (B, \mu, \Delta, \eta, \epsilon)$ is a convolution invertible element $\rho \in (B \otimes B)^*$ such that

(1) $\rho * \mu = \mu^{\mathrm{op}} * \rho$,
(2) $\rho_{1,3} * \rho_{1,2} = \rho(\mathrm{id}_B \otimes \mu)$, and
(3) $\rho_{1,3} * \rho_{2,3} = \rho(\mu \otimes \mathrm{id}_B)$.

A bialgebra provided with a dual universal R-matrix is called *cobraided.*

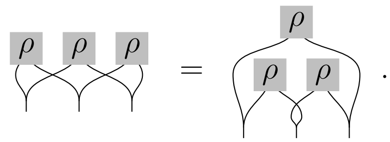
In terms of string diagrams,



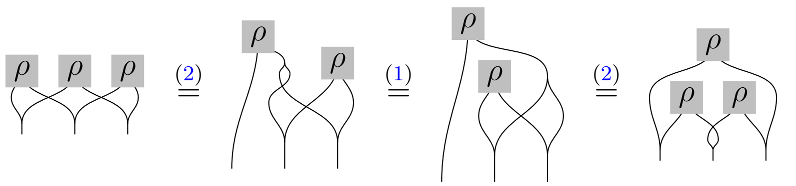
Proposition 21. *A dual universal R-matrix in a bialgebra B satisfies the following* Yang-Baxter relation *in the convolution algebra $(B^{\otimes 3})^*$:*

(3) $$\rho_{1,2} * \rho_{1,3} * \rho_{2,3} = \rho_{2,3} * \rho_{1,3} * \rho_{1,2}.$$

That is, in terms of string diagrams,



Proof. The proof is exactly the mirror image with respect to the horizontal axis of the proof of the first identity in Proposition 19:



□

Exercise 8. State and prove the properties of the dual universal R-matrix analogous to the (2) and (3) of Proposition 19.

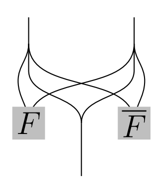
10. Lecture 10 (Thu Feb 22, 2024)

10.1. **Quantum double.** Drinfeld's quantum double construction is a way of producing braided Hopf algebras. We mostly follow [KRT97, Ch. 3], [ES98, Ch. 12], and [Kas23, Ch. 5].

10.1.1. *Bialgebras twisted by cycles.* Let $B = (B, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. Choose an invertible element $F \in B \otimes B$ and set

$$\begin{aligned} \Delta_F : B &\to B \otimes B \\ x &\mapsto F\Delta(x)F^{-1}, \end{aligned}$$

i.e.

$\Delta_F \;:=$  ,

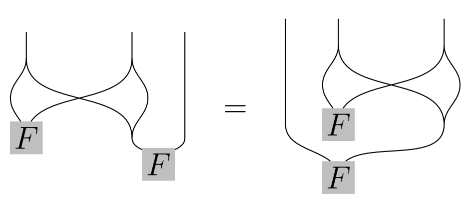
where $\overline{F}$ denotes F^{-1}.

This is clearly an algebra homomorphism. The following proposition gives a sufficient condition for Δ_F to be a coproduct with counit ϵ:

Proposition 22. *(1) If F satisfies*

$$F_{12} \cdot (\Delta \otimes \mathrm{id}_B)(F) = F_{23} \cdot (\mathrm{id}_B \otimes \Delta)F,$$

i.e.

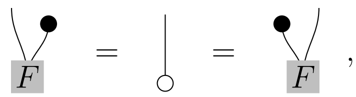


then Δ_F is coassociative.

(2) If F satisfies

$$(\mathrm{id}_B \otimes \epsilon)(F) = 1 = (\epsilon \otimes \mathrm{id}_B)(F),$$

i.e.



then ϵ is a counit with respect to Δ_F.

Proof. (1) For any $x \in B$,

$$\begin{aligned}((\Delta_F \otimes \mathrm{id}_B)\Delta_F)(x) &= (\Delta_F \otimes \mathrm{id}_B)(F\Delta(x)F^{-1}) \\ &= (\Delta_F \otimes \mathrm{id}_B)(F) \cdot (\Delta_F \otimes \mathrm{id}_B)(\Delta(x)) \cdot (\Delta_F \otimes \mathrm{id}_B)(F^{-1}) \\ &= F_{12} \cdot (\Delta \otimes \mathrm{id}_B)(F) \cdot (\Delta \otimes \mathrm{id}_B)(\Delta(x)) \cdot (\Delta \otimes \mathrm{id}_B)(F^{-1}) \cdot F_{12}^{-1},\end{aligned}$$

and likewise

$$((\mathrm{id}_B \otimes \Delta)\Delta_F)(x) = F_{23} \cdot (\mathrm{id}_B \otimes \Delta)(F) \cdot (\mathrm{id}_B \otimes \Delta)(\Delta(x)) \cdot (\mathrm{id}_B \otimes \Delta)(F^{-1}) \cdot F_{23}^{-1}.$$

Therefore, Δ_F would be coassociative if

$$F_{12} \cdot (\Delta \otimes \mathrm{id}_B)(F) = F_{23} \cdot (\mathrm{id}_B \otimes \Delta)F.$$

(2) For any $x \in B$,

$$\begin{aligned}(\mathrm{id}_B \otimes \epsilon)(\Delta_F(x)) &= (\mathrm{id}_B \otimes \epsilon)(F\Delta(x)F^{-1}) \\ &= (\mathrm{id}_B \otimes \epsilon)(F) \cdot x \cdot (\mathrm{id}_B \otimes \epsilon)(F^{-1}),\end{aligned}$$

and this would be equal to x if

$$(\mathrm{id}_B \otimes \epsilon)(F) = 1.$$

Likewise, we would have $(\mathrm{id}_B \otimes \epsilon)(\Delta_F(x)) = x$ if $(\epsilon \otimes \mathrm{id}_B)(F) = 1$.

□

Definition 34. A *cycle* in a bialgebra $B = (B, \mu, \eta, \Delta, \epsilon)$ is an invertible element $F \in B \otimes B$ satisfying the two conditions in Proposition 22.

Then, $B_F := (B, \mu, \eta, \Delta_F, \epsilon)$ with the twisted coproduct

$$\Delta_F := F * \Delta * F^{-1} \in L(B, B \otimes B)$$

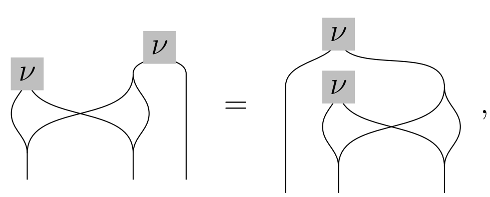
is called the *bialgebra twisted by cycle F*.

For later purposes, we record the dual notions here.

Definition 35. A *cocycle* in a bialgebra $B = (B, \mu, \eta, \Delta, \epsilon)$ is an invertible element ν in the convolution algebra $(B \otimes B)^*$ such that

$$\nu((\nu * \mu) \otimes \mathrm{id}_B) = \nu(\mathrm{id}_B \otimes (\nu * \mu)),$$

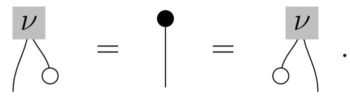
i.e.



and

$$\nu(\eta \otimes \mathrm{id}_B) = \epsilon = \nu(\mathrm{id}_B \otimes \eta),$$

i.e.



Proposition 23. *Let $B = (B, \mu, \eta, \Delta, \epsilon)$ be a bialgebra and ν a cocycle in B. Then, $B_\nu := (B, \mu_\nu, \eta, \Delta, \epsilon)$ with the twisted product*

$$\mu_\nu := \nu * \mu * \overline{\nu}$$

is a bialgebra.

10.1.2. *Dual double construction.*

Proposition 24. *Let $H = (H, \mu, \eta, \Delta, \epsilon, S)$ be a finite dimensional Hopf algebra with invertible antipode S. Define $\widetilde{H} := H^* \otimes H^{\mathrm{op}}$. Let $\{e_i\}_{i\in I}$ be a basis of H and $\{e^i\}_{i\in I}$ a dual basis of H^*. Then, the canonical element*

$$\widetilde{F} := \sum_{i\in I} (1_{H^*} \otimes e_i) \otimes (e^i \otimes 1_H) \ \in \ \widetilde{H} \otimes \widetilde{H}$$

is invertible, and $\widetilde{F}^{-1}$ is a cycle in $\widetilde{H}$.

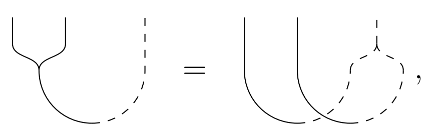
Proof. As an algebra, $H^* \otimes H^{\mathrm{op}}$ can be canonically identified with $\mathrm{End}(H^{\mathrm{op}})$ with the convolution product. The canonical element $\sum_{i\in I} e_i \otimes e^i$ corresponds to $\mathrm{id}_{H^{\mathrm{op}}} \in \mathrm{End}(H^{\mathrm{op}})$ under this identification, and its inverse is the antipode S^{-1} of H^{op}. From this, we see that $\widetilde{F}$ is invertible with inverse given by

$$\widetilde{F}^{-1} = \sum_{i\in I} (1_{H^*} \otimes S^{-1} e_i) \otimes (e^i \otimes 1_H).$$

Now let's show that $\widetilde{F}^{-1}$ is a cycle. Since the coproduct (resp. product) in H^* is the transpose of the product (resp. coproduct) in H, we have

$$(\Delta \otimes \mathrm{id}_H)(\widetilde{F}) = \widetilde{F}_{13}\widetilde{F}_{23},$$

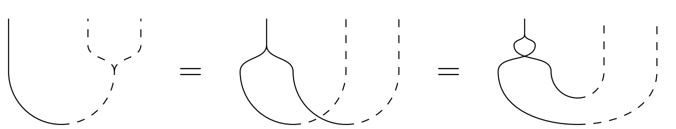
i.e.



and

$$(\mathrm{id}_H \otimes \Delta)(\widetilde{F}) = \widetilde{F}_{13}\widetilde{F}_{12},$$

i.e.



Since $\widetilde{F}_{12}$ and $\widetilde{F}_{23}$ commute,

$$\begin{aligned}(\Delta \otimes \mathrm{id}_H)(\widetilde{F}) \cdot \widetilde{F}_{12} &= \widetilde{F}_{13}\widetilde{F}_{23}\widetilde{F}_{12} \\ &= \widetilde{F}_{13}\widetilde{F}_{12}\widetilde{F}_{23} \\ &= (\mathrm{id}_H \otimes \Delta)(\widetilde{F}) \cdot \widetilde{F}_{23}.\end{aligned}$$

Taking the inverse, we get

$$\widetilde{F}_{12}^{-1} \cdot (\Delta \otimes \mathrm{id}_H)(\widetilde{F}^{-1}) = \widetilde{F}_{23}^{-1} \cdot (\mathrm{id}_H \otimes \Delta)(\widetilde{F}^{-1}),$$

which is condition (1) of a cycle.

For condition (2), we can take 1_H to be one of the basis vectors, e_1, and take the remaining ones in $\ker \epsilon$. Then,

$$\begin{aligned}(\mathrm{id}_{\widetilde H}\otimes\epsilon_{\widetilde H})(\widetilde F^{-1}) &= (\mathrm{id}_{\widetilde H}\otimes(\eta^*\otimes\epsilon))\left(\sum_{i\in I}(1_{H^*}\otimes S^{-1}e_i)\otimes(e^i\otimes 1_H)\right)\\ &= \sum_{i\in I}(1_{H^*}\otimes S^{-1}e_i)\otimes(\eta^*(e^i)\otimes\epsilon(1_H))\\ &= 1_{H^*}\otimes S^{-1}1_H\\ &= 1_{H^*}\otimes 1_H = 1_{\widetilde H},\end{aligned}$$

and likewise

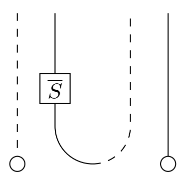
$$\begin{aligned}(\epsilon_{\widetilde H}\otimes\mathrm{id}_{\widetilde H})(\widetilde F^{-1}) &= \sum_{i\in I}(\eta^*(1_{H^*})\otimes\epsilon(S^{-1}e_i))\otimes(e^i\otimes 1_H)\\ &= 1_{H^*}\otimes 1_H = 1_{\widetilde H}.\end{aligned}$$

□

Theorem 14. *In the setup of Proposition 24, the bialgebra $\widetilde H_F = H^*\otimes H^{\mathrm{op}}$ twisted by cycle*

$$F := \widetilde F^{-1} = \sum_{i\in I}(1_{H^*}\otimes S^{-1}e_i)\otimes(e^i\otimes 1_H),$$

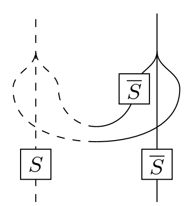
i.e.

$F =$  $\in \widetilde H_F\otimes\widetilde H_F,$

is a Hopf algebra with invertible antipode given by

$$\begin{aligned}S_{\widetilde H_F} : \widetilde H_F &\to \widetilde H_F\\ \ell\otimes x &\mapsto f\big(S_{H^*}(\ell)\otimes S_H^{-1}(x)\big)f^{-1},\end{aligned}$$

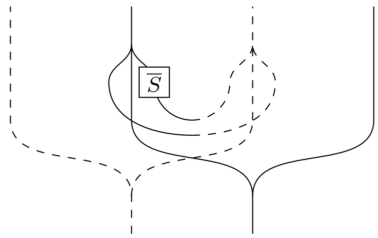
where $f = \sum_{i\in I} e^i\otimes e_i \in \widetilde H_F$, i.e.

$S_{\widetilde H_F} =$  $\in L(\widetilde H_F, \widetilde H_F).$

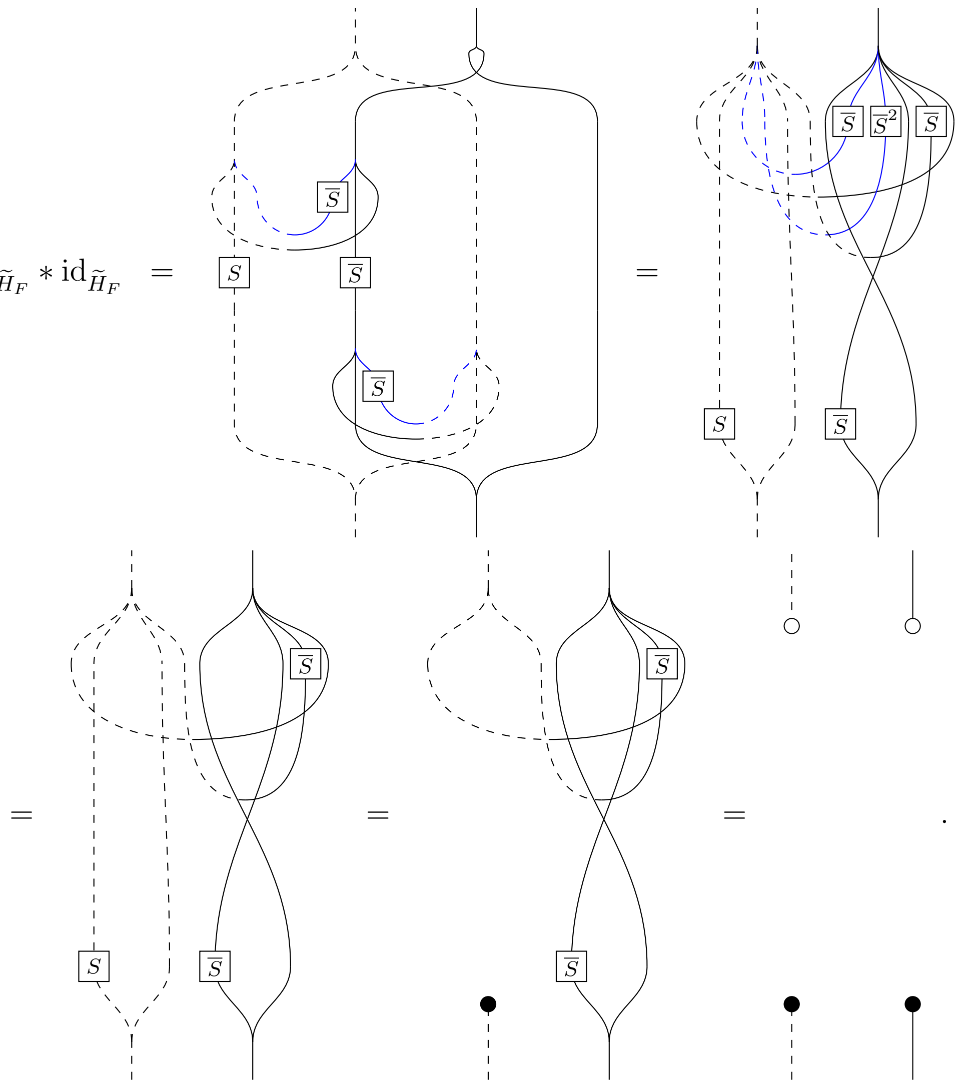
Proof. Invertibility of $S_{\widetilde H_F}$ follows from that of S_H, so it suffices to show that, for all $\ell\in H^*$ and $x\in H$,

$$(\mu_{\widetilde H_F}\circ(S_{\widetilde H_F}\otimes\mathrm{id}_{\widetilde H_F}))\Delta_F(\ell\otimes x) = \eta_{\widetilde H_F}\circ\epsilon_{\widetilde H_F}(\ell\otimes x) = (\mu_{\widetilde H_F}\circ(\mathrm{id}_{\widetilde H_F}\otimes S_{\widetilde H_F}))\Delta_F(\ell\otimes x).$$

We will give a diagrammatic proof. Note, the twisted coproduct Δ_F is given by

$\Delta_F =$  $\in L(\widetilde{H}_F, \widetilde{H}_F)$.

So, we have

$S_{\widetilde{H}_F} * \mathrm{id}_{\widetilde{H}_F} =$ 

The other relation can be proved in a similar way. □

10.1.3. *Quantum double.* In this subsection, we will see that the Hopf algebra $\widetilde{H}_F$ is actually a cobraided Hopf algebra (or equivalently, that its dual Hopf algebra is a braided Hopf algebra).

Definition 36 (Quantum double)**.** Let H be a finite dimensional Hopf algebra with invertible antipode S. The *quantum double* $D(H) = H \otimes H^{*,\mathrm{cop}}$ of a Hopf algebra H is the dual Hopf algebra of $\widetilde{H}_F$ constructed in Theorem 14.

Theorem 15. *The quantum double $D(H)$ has the following properties:*

(1) As a coalgebra, it is the tensor product of coalgebras H and $H^{*,\mathrm{cop}}$.
(2) Via the natural inclusions

$$\begin{aligned} H &\to D(H) \\ x &\mapsto x \otimes 1 \end{aligned}$$

and

$$\begin{aligned} H^{*,\mathrm{cop}} &\to D(H) \\ \ell &\mapsto 1 \otimes \ell, \end{aligned}$$

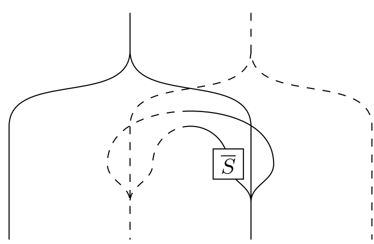
H and $H^{*,\mathrm{cop}}$ become Hopf subalgebras of $D(H)$.
(3) For all $x \in H$ and $\ell \in H^*$, we have

$$(x \otimes 1)(1 \otimes \ell) = x \otimes \ell$$

and

$$(1 \otimes \ell)(x \otimes 1) = \sum_{(x),(\ell)} \langle \ell_{(1)}, S^{-1}x_{(1)} \rangle \langle \ell_{(3)}, a_{(3)} \rangle a_{(2)} \otimes \ell_{(2)},$$

i.e.

$\mu_{D(H)} =$  $\in L(D(H) \otimes D(H), D(H)).$

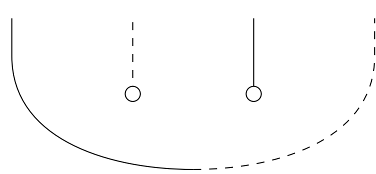
Proof. (1) Obvious.
(2) We just need to check that these inclusions are algebra maps. We leave this as an exercise.
(3) This follows from the definition of the twisted coproduct Δ_F. We leave this as an exercise as well.

∎

Theorem 16. *Let $R \in D(H) \otimes D(H)$ be the canonical element*

$$R = \sum_{i \in I} (e_i \otimes 1) \otimes (1 \otimes e^i) \in D(H) \otimes D(H),$$

i.e.

$R =$  .

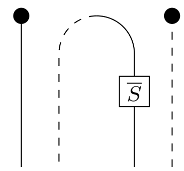
Then, R is a universal R-matrix for $D(H)$, with inverse

$$R^{-1} = \sum_{i \in I} (Se_i \otimes 1) \otimes (1 \otimes e^i).$$

10.1.4. *Quantum doubles for infinite dimensional Hopf algebras.* The quantum double construction can be stated more generally for infinite dimensional Hopf algebras as well, by using the restricted dual. In this setup, we get a bialgebra $D(H)$ whose restricted dual $D(H)^o$ is cobraided:

Theorem 17 (Quantum double, more general setting)**.** *Let H be a Hopf algebra with invertible antipode S. Let $D(H)$ be the bialgebra $H \otimes H^{o,\mathrm{cop}}$ twisted by the cocycle*

$$\nu_H = \epsilon_H \otimes \mathrm{ev}_H(\mathrm{id}_{H^o} \otimes S^{-1}) \otimes \epsilon_{H^o} =$$



.

Then it is a Hopf algebra containing H and $H^{o,\mathrm{cop}}$ as Hopf subalgebras through the following canonical bialgebra embeddings:

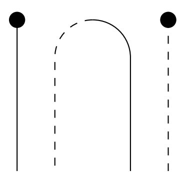
$$\begin{aligned} \iota : H &\hookrightarrow D(H) \\ x &\mapsto x \otimes 1_{H^o} \end{aligned}$$

$$\begin{aligned} \jmath : H^{o,\mathrm{cop}} &\hookrightarrow D(H) \\ \ell &\mapsto 1_H \otimes \ell. \end{aligned}$$

Moreover, $D(H)^o$ is cobraided *with the dual universal R-matrix*

$$\rho := \mathrm{ev}_{D(H)}((\jmath\, \iota^o) \otimes \mathrm{id}_{D(H)^o}) \ \in \ (D(H)^o \otimes D(H)^o)^*.$$

Remark 12. While we have been using $H \otimes H^{o,\mathrm{cop}}$ following [KRT97], in some literature (e.g. [Kas23]), the quantum double $D(H)$ is defined as a bialgebra $H \otimes H^{o,\mathrm{op}}$, twisted by the cocycle $\epsilon_H \otimes \mathrm{ev}_H \otimes \epsilon_{H^o}$:



.

Then, $D(H)^o$ is cobraided with the dual universal R-matrix

$$\rho := \mathrm{ev}_{D(H)}(\mathrm{id}_{D(H)^o} \otimes (\jmath\, \iota^o)) \ \in \ (D(H)^o \otimes D(H)^o)^*.$$

This is of course a different definition, but at least for finite dimensional Hopf algebras, the two conventions are related by

$$\begin{aligned} H &\leftrightarrow H^{*,\mathrm{op}} \\ H^{*,\mathrm{cop}} &\leftrightarrow H. \end{aligned}$$

One advantage of the latter convention is that, even if the antipode S of H is not invertible, we still get a bialgebra $D(H)$ whose restricted dual $D(H)^o$ is cobraided. If S is invertible, then $D(H)$ is a Hopf algebra.

Remark 13. When H is finite dimensional, that $D(H)^*$ is cobraided just means that $D(H)$ has a universal R-matrix. More explicitly, if $\{e_i\}_{i\in I}$ is a linear basis of H and $\{e^i\}_{i\in I}$ is a dual basis of H^*, then the dual universal R-matrix $\rho \in (D(H)^o \otimes D(H)^o)^*$ is conjugate to the universal R-matrix

$$R := \sum_{i\in I} \iota e_i \otimes \jmath e^i \in D(H) \otimes D(H)$$

in the sense that, for any $x, y \in D(H)^o = D(H)^*$,

$$\langle x \otimes y, R\rangle = \sum_{i\in I}\langle x, \iota e_i\rangle\langle y, \jmath e^i\rangle = \sum_{i\in I}\langle \iota^o x, e_i\rangle\langle y, \jmath e^i\rangle$$
$$= \left\langle y,\ \jmath\left(\sum_{i\in I}\langle \iota^o x, e_i\rangle e^i\right)\right\rangle = \langle y, \jmath\iota^o x\rangle = \langle \rho, x\otimes y\rangle.$$

However, when H is infinite dimensional, the expression of R above is only formal; the universal R-matrix lives in some completion of $D(H)\otimes D(H)$. In view of the algebra homomorphism

$$D(H)\otimes D(H) \to (D(H)^o \otimes D(H)^o)^*,$$

we can think of $(D(H)^o\otimes D(H)^o)^*$ as a certain algebra completion of $D(H)\otimes D(H)$.

11. Lecture 11 (Tue Feb 27, 2024)

11.1. **Quantum double (cont.)**

Proof of Theorem 16. The only non-trivial thing to prove is condition (1) – that $R\Delta(h) = \Delta^{\mathrm{op}}(h)R$ for all $h \in D(H)$. Since the subspace

$$\{h \in D(H) \mid R\Delta(h) = \Delta^{\mathrm{op}}(h)R\}$$

is a subalgebra of $D(H)$, we just need to check this relation for $h = x\otimes 1$ with $x \in H$ and $h = 1\otimes \ell$ with $\ell \in H^*$.

The following lemma, whose proof is an easy exercise, will be useful:

Lemma 3. *Let H be a Hopf algebra and $R = \sum_i \alpha_i \otimes \beta_i \in H\otimes H$ be an invertible element. Then, the followings are equivalent:*

(i) $R\Delta(h) = \Delta^{\mathrm{op}}(h)R$, *for all* $h\in H$.
(ii) $R(h\otimes 1) = \sum_{(h)} h_{(2)}\alpha_i \otimes h_{(1)}\beta_i S h_{(3)}$, *for all* $h \in H$.
(iii) $(h\otimes 1)R = \sum_{(h)} \alpha_i h_{(2)} \otimes S h_{(1)}\beta_i h_{(3)}$, *for all* $h\in H$.
(iv) $R(1\otimes h) = \sum_{(h)} h_{(3)}\alpha_i S^{-1}h_{(1)} \otimes h_{(2)}\beta_i$, *for all* $h \in H$.
(v) $(1\otimes h)R = \sum_{(h)} S^{-1}h_{(3)}\alpha_i h_{(1)} \otimes \beta_i h_{(2)}$, *for all* $h\in H$.

Now, let's check the condition (iv) for elements $x\otimes 1$ and condition (iii) for elements $1\otimes \ell$. The LHS of (iv) with $h = x\otimes 1$ is equal to

$$R((1\otimes 1)\otimes(x\otimes 1)) = \sum_{i\in I}(e_i\otimes 1)\otimes\big((1\otimes e^i)(x\otimes 1)\big)$$
$$= \sum_{\substack{i\in I\\(x)(e^i)}} (e_i\otimes 1)\otimes(x_{(2)}\otimes e^i_{(2)})\langle e^i_{(1)}, S^{-1}x_{(1)}\rangle\langle e^i_{(3)}, x_{(3)}\rangle,$$

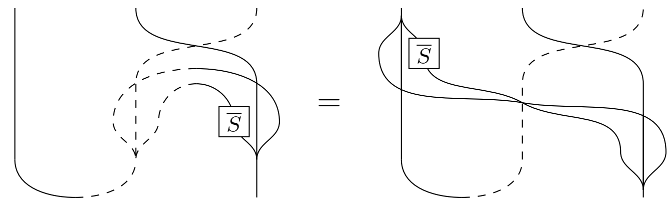
while the RHS is equal to

$$\sum_{\substack{j\in I\\(x)}} \big((x_{(3)}e_j S^{-1}x_{(1)})\otimes 1\big)\otimes(x_{(2)}\otimes e^j)$$
$$= \sum_{\substack{i,j\in I\\(x)}} (e_i\otimes 1)\otimes(x_{(2)}\otimes e^j)\langle e^i, x_{(3)}e_j S^{-1}x_{(1)}\rangle$$

$$= \sum_{\substack{i,j\in I\\(x)(e^i)}} (e_i\otimes 1)\otimes(x_{(2)}\otimes e^j)\langle e^i_{(1)}, S^{-1}x_{(1)}\rangle\langle e^i_{(2)}e_j\rangle\langle e^i_{(3)}, x_{(3)}\rangle$$

$$= \sum_{\substack{i\in I\\(x)(e^i)}} (e_i\otimes 1)\otimes(x_{(2)}\otimes e^i_{(2)})\langle e^i_{(1)}, S^{-1}x_{(1)}\rangle\langle e^i_{(3)}, x_{(3)}\rangle,$$

i.e.

LHS =  = RHS.

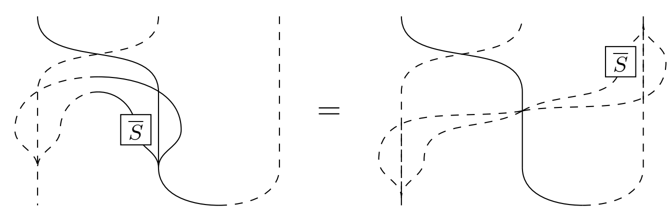
Likewise, the LHS of (iii) with $h = 1\otimes\ell$ is equal to

$$((1\otimes\ell)\otimes(1\otimes 1))R = \sum_{i\in I}((1\otimes\ell)(e_i\otimes 1))\otimes(1\otimes e^i)$$
$$= \sum_{\substack{i\in I\\(e_i)(\ell)}} \langle \ell_{(1)}, S^{-1}e_{i(1)}\rangle\langle \ell_{(3)}, e_{i(3)}\rangle(e_{i(2)}\otimes\ell_{(2)})\otimes(1\otimes e^i),$$

while the RHS is equal to

$$\sum_{\substack{j\in I\\(\ell)}}(e_j\otimes\ell_{(2)})\otimes(1\otimes S^{-1}\ell_{(1)}e^j\ell_{(3)})$$
$$= \sum_{\substack{i,j\in I\\(\ell)}}\langle S^{-1}\ell_{(1)}e^j\ell_{(3)}, e_i\rangle(e_j\otimes\ell_{(2)})\otimes(1\otimes e_i)$$
$$= \sum_{\substack{i,j\in I\\(\ell)}}\langle S^{-1}\ell_{(1)}, e_{i(1)}\rangle\langle e^j, e_{i(2)}\rangle\langle\ell_{(3)}, e_{i(3)}\rangle(e_j\otimes\ell_{(2)})\otimes(1\otimes e_i)$$
$$= \sum_{\substack{i\in I\\(e_i)(\ell)}} \langle \ell_{(1)}, S^{-1}e_{i(1)}\rangle\langle \ell_{(3)}, e_{i(3)}\rangle(e_{i(2)}\otimes\ell_{(2)})\otimes(1\otimes e^i),$$

i.e.

LHS =  = RHS.

□

Therefore, $D(H)$ is braided. In fact, we could have characterized $D(H)$ from the property that the canonical element R is a universal R-matrix for $D(H)$:

Proposition 25. *The algebra structure in $D(H)$ is the only Hopf algebra structure on the coalgebra $H\otimes H^{*,\mathrm{cop}}$ such that*

(1) H and $H^{*,\mathrm{cop}}$ are Hopf subalgebras,

(2) the multiplication map

$$H \otimes H^{*,\mathrm{cop}} \to D(H)$$
$$x \otimes \ell \mapsto x \cdot \ell$$

is an isomorphism of vector spaces,

(3) and $R = \sum_{i \in I} e_i \otimes e^i \in D(H) \otimes D(H)$ *is a universal R-matrix.*

Proof. The only structure to be determined is products of the form $\ell \cdot x$ with $\ell \in H^{*,\mathrm{cop}}$ and $x \in H$. Suppose that $R = \sum_{i \in I} e_i \otimes e^i$ is a universal R-matrix for $D(H)$. Then, it should satisfy the Yang-Baxter relation (Proposition 19)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

That is,

$$\sum_{i,j,k \in I} e_i e_j \otimes e^i e_k \otimes e^j e^k = \sum_{i,j,k \in I} e_j e_i \otimes e_k e^i \otimes e^k e^j.$$

Note that the LHS contains products of the form $e^i e_k$. We can thus view the above as a system of linear equations in the unknowns $e^i e_k$.

We claim that the vectors $v_{ik} := \sum_j e_i e_j \otimes e^j e^k$ form a basis of $H \otimes H^*$. Then it would immediately follow that the products $e^i e_k$ are uniquely determined by the Yang-Baxter equation. For this, note that

$$v_{ik} = (e_i \otimes e^k)(\sum_j e_j \otimes e^j) = (e_i \otimes e^k)R$$

in $H^{\mathrm{cop}} \otimes (H^{\mathrm{cop}})^*$. Therefore, it suffices to show that R is invertible in $H^{\mathrm{cop}} \otimes (H^{\mathrm{cop}})^*$. This follows from

$$R\,(\mathrm{id}_H \otimes S^{-1})(R) = 1 = (\mathrm{id}_H \otimes S^{-1})(R)\,R.$$

□

Example 12. Let G be a finite group and let $H = \mathbb{C}[G]$ be the group algebra. Then $H^{*,\mathrm{cop}} = (\mathbb{C}^G)^{\mathrm{cop}}$ is the commutative algebra of functions on G with coproduct given by

$$\Delta(f)(x \otimes y) = f(yx).$$

The quantum double is the Hopf algebra $D(H) = \mathbb{C}[G] \otimes (\mathbb{C}^G)^{\mathrm{cop}}$, with the product determined by the relation

$$\begin{aligned} f(x) \cdot g &= \sum_{(g),(f)} f_{(1)}(S^{-1}g_{(1)}) f_{(3)}(g_{(3)})\; g_{(2)} \cdot f_{(2)}(x) \\ &= \sum_{(f)} f_{(1)}(g^{-1}) f_{(3)}(g)\; g \cdot f_{(2)}(x) \\ &= g \cdot \Delta^{(2)}(f)(g^{-1} \otimes x \otimes g) \\ &= g \cdot f(gxg^{-1}). \end{aligned}$$

That is, as an algebra, $D(H)$ is the semidirect product $\mathbb{C}[G] \ltimes (\mathbb{C}^G)^{\mathrm{cop}}$.

The universal R-matrix is given by

$$R = \sum_{x \in G} x \otimes \delta_x,$$

with inverse

$$R^{-1} = \sum_{x\in G} x^{-1} \otimes \delta_x.$$

We can easily double check that it intertwines the coproduct:

$$\begin{aligned}
R\Delta(g\delta_h) &= \left(\sum_{x\in G} x \otimes \delta_x\right)\left(\sum_{y\in G} g\delta_y \otimes g\delta_{hy^{-1}}\right) \\
&= \sum_{x,y\in G} xg\delta_y \otimes \delta_x g\delta_{hy^{-1}} \\
&= \sum_{x,y\in G} xg\delta_y \otimes g\delta_{g^{-1}xg}\delta_{hy^{-1}} \\
&= \sum_{x',y\in G} gx'\delta_y \otimes g\delta_{x'}\delta_{hy^{-1}} \\
&= \sum_{x',y\in G} g\delta_{x'yx'^{-1}}x' \otimes g\delta_{x'}\delta_{hy^{-1}} \\
&= \sum_{x',y\in G} g\delta_{hyh^{-1}}x' \otimes g\delta_{x'}\delta_{hy^{-1}} \\
&= \sum_{x',y'\in G} g\delta_{hy'^{-1}}x' \otimes g\delta_{x'}\delta_{y'} \\
&= \left(\sum_{y'\in G} g\delta_{hy'^{-1}} \otimes g\delta_{y'}\right)\left(\sum_{x'\in G} x' \otimes \delta_{x'}\right) \\
&= \Delta^{\mathrm{op}}(g\delta_h)R
\end{aligned}$$

The braided Hopf algebra $D(H)$ is in fact ribbon, with a ribbon element given by

$$u = \sum_{x\in G} x^{-1}\delta_x,$$

with inverse

$$\theta = u^{-1} = \sum_{x\in G} x\delta_x.$$

We leave it as an exercise to check that this is indeed a ribbon element.

Remark 14. The construction of quantum double presented here was purely formal and may seem complicated and unmotivated. However, there is a much nicer way to think about it, in terms of line operators in the bulk of a 3d TQFT and on two transverse boundary theories. This involves extended TQFTs, so I wouldn't have time to cover this in this course, but I highly recommend checking out Tudor Dimofte's talk at Perimeter Institute, titled "Spark algebras and quantum groups".

11.2. **Quantum groups.** We follow [KRT97, Ch. 4].

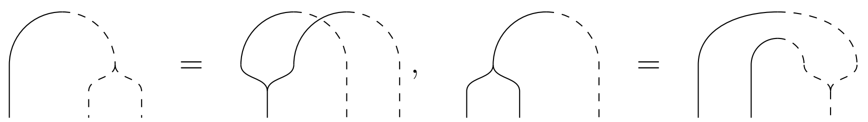
11.2.1. *Hopf pairing and generalized double.*

Definition 37. Let A and B be Hopf algebras over $\Bbbk$ with invertible antipodes. A *Hopf pairing* between A and B is a bilinear form $\varphi : A \otimes B \to \Bbbk$ such that

$$\begin{aligned} \varphi(a, bb') &= \sum_{(a)} \varphi(a_{(1)}, b)\varphi(a_{(2)}, b'), \\ \varphi(aa', a) &= \sum_{(b)} \varphi(a, b_{(2)})\varphi(a', b_{(1)}), \\ \varphi(a, 1_B) &= \epsilon_A(a) \quad \text{and} \quad \varphi(1_A, b) = \epsilon_B(b), \\ \varphi(Sa, b) &= \varphi(a, S^{-1}b) \end{aligned}$$

for all $a, a' \in A$ and $b, b' \in B$.

In other words, if we draw A as solid lines and B as dashed lines,



etc.

Example 13. For any Hopf algebra H, the natural evaluation map

$$\mathrm{ev}_H : H \otimes H^{o,\mathrm{cop}} \to \Bbbk$$

is a Hopf pairing.

Theorem 18. *Given a Hopf pairing $\varphi : A \otimes B \to \Bbbk$, there is a unique Hopf algebra structure on $A \otimes B$ satisfying conditions in Theorem 15, with H and $H^{*,\mathrm{cop}}$ replaced by A and B, respectively.*

The proof is exactly the same as before. The resulting Hopf algebra is denoted by $D_\varphi(A, B)$ and is called the *generalized double* of A with respect to B and φ.

Remark 15. While $D_\varphi(A, B)$ is a Hopf algebra, it doesn't have to be braided. If A and B are finite dimensional Hopf algebras and the pairing φ is non-degenerate, then by the same construction as before, $D_\varphi(A, B)$ becomes a braided Hopf algebra. If φ is degenerate, then we can quotient A and B out by the annihilator ideals

$$\begin{aligned} I_A &:= \{a \in A \mid \varphi(a, b) = 0 \text{ for all } b \in B\}, \\ I_B &:= \{b \in B \mid \varphi(a, b) = 0 \text{ for all } a \in A\} \end{aligned}$$

to get a non-degenerate Hopf pairing

$$\overline{\varphi} : A/I_A \otimes B/I_B \to \Bbbk,$$

which gives rise to a braided Hopf algebra $D_{\overline{\varphi}}(A/I_A, B/I_B)$.

The following lemma will be useful later:

Lemma 4. *Let $\widetilde{A}$ and $\widetilde{B}$ be free algebras generated by elements $a_1, \cdots, a_m$ and $b_1, \cdots, b_n$, respectively. Let $\{\lambda_{ij}\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ be mn scalars. Then, there is a unique Hopf pairing $\varphi : \widetilde{A} \otimes \widetilde{B} \to \Bbbk$ such that $\varphi(a_i, b_j) = \lambda_{ij}$.*

Moreover, if A (resp. B) is a quotient of $\widetilde{A}$ (resp. $\widetilde{B}$) by the ideal generated by elements $r_1, \cdots, r_k$ (resp. $s_1, \cdots, s_l$), then the Hopf pairing $\varphi : \widetilde{A} \otimes \widetilde{B} \to \Bbbk$ descends to a Hopf pairing $\varphi : A \otimes B \to \Bbbk$ if and only if $\varphi(r_i, b_j) = 0$ for all $1 \le i \le k$, $1 \le j \le n$ and $\varphi(a_i, s_j) = 0$ for all $1 \le i \le m$, $1 \le j \le l$.

11.2.2. *Brief review of* $U(\mathfrak{sl}_{N+1})$. Let $\mathfrak{sl}_{N+1}$ be the Lie algebra of traceless complex $(N+1)\times(N+1)$ matrices. Let $e_{i,j}$ be the elementary matrix whose entries are all 0 except for the (i,j)-entry which is 1. Then,

$$\{e_{i,j} \mid 1 \le i,j \le N+1,\ i \ne j\} \cup \{e_{i,i} - e_{i+1,i+1} \mid 1 \le i \le N\}$$

form a basis of $\mathfrak{sl}_{N+1}$. For $1 \le i \le N$, let

$$E_i := e_{i,i+1}, \quad F_i := e_{i+1,i}, \quad H_i := e_{i,i} - e_{i+1,i+1}.$$

The *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{sl}_{N+1}$ is the Lie subalgebra generated by traceless diagonal matrices $H_1, \cdots, H_N$. Let ϵ be a linear form on the space of diagonal matrices defined by

$$\epsilon_i(e_{j,j}) = \delta_{i,j}.$$

Then, the *simple roots*,

$$\{\alpha_i = \epsilon_i - \epsilon_{i+1}\}_{1 \le i \le N},$$

forms a basis of $\mathfrak{h}^*$. The matrix

$$(\alpha_j(H_i)) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

is called the *Cartan matrix* of $\mathfrak{sl}_{N+1}$.

The E_i, F_i, H_i $(1 \le i \le N)$ generate the Lie algebra $\mathfrak{sl}_{N+1}$ (and its universal enveloping algebra $U(\mathfrak{sl}_{N+1})$) subject to relations

$$[H_i, H_j] = 0, \quad [H_i, E_j] = \alpha_j(H_i)E_j, \quad [H_i, F_j] = -\alpha_j(H_i)F_j, \quad [E_i, F_j] = \delta_{ij}H_i, \quad 1 \le i,j \le N$$

$$[E_i, E_j] = 0, \quad [F_i, F_j] = 0, \quad \text{if } |i-j| \ge 2,$$

$$[E_i, [E_i, E_j]] = 0, \quad [F_i, [F_i, F_j]] = 0, \quad \text{if } |i-j| = 1.$$

The adjoint action of $\mathfrak{h}$ makes $U(\mathfrak{sl}_{N+1})$ into a Q-graded algebra, where Q is the *root lattice* (i.e. the free abelian group with basis $\alpha_1, \cdots, \alpha_N$). The degree of the generators are given by

$$\deg E_i = \alpha_i, \quad \deg F_i = -\alpha_i, \quad \deg H_i = 0.$$

12. Lecture 12 (Thu Feb 29, 2024)

12.1. **Quantum groups (cont.)**

12.1.1. *Quantized enveloping algebras* $U_q(\mathfrak{sl}_{N+1})$. Let $\widetilde{U}_+$ be the $\mathbb{C}(q)$-algebra generated by E_i, $K_i^{\pm 1}$ $(1 \le i \le N)$, subject to the relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \quad K_iE_j = q^{\alpha_j(H_i)}E_jK_i.$$

Simiarly, let $\widetilde{U}_-$ be the $\mathbb{C}(q)$-algebra generated by F_i, ${K'}_i^{\pm 1}$ $(1 \le i \le N)$, subject to the relations

$$K'_i{K'}_i^{-1} = {K'}_i^{-1}K'_i = 1, \quad K'_iK'_j = K'_jK'_i, \quad K'_iF_j = q^{-\alpha_j(H_i)}F_jK'_i.$$

The algebras $\widetilde{U}_+$ and $\widetilde{U}_-$ have Hopf algebra structures given by

$$\Delta K_i = K_i \otimes K_i, \quad \Delta E_i = E_i \otimes 1 + K_i \otimes E_i,$$

$$\Delta K'_i = K'_i \otimes K'_i, \quad \Delta F_i = F_i \otimes {K'}_i^{-1} + 1 \otimes F_i.$$

They are Q-graded, with

$$\deg E_i = \alpha_i, \quad \deg F_i = -\alpha_i, \quad \deg K_i = \deg K_i' = 0.$$

The following theorem follows from Lemma 4:

Theorem 19. *There exists a unique Hopf pairing*

$$\varphi : \widetilde{U}_+ \times \widetilde{U}_- \to \mathbb{C}(q)$$

such that

$$\varphi(E_i, F_j) = -\frac{\delta_{ij}}{q - q^{-1}},$$
$$\varphi(E_i, K_j') = \varphi(K_i, F_j) = 0,$$
$$\varphi(K_i, K_j') = q^{-\alpha_i(H_j)} = q^{-\alpha_j(H_i)}.$$

The resulting Hopf algebra $D(\widetilde{U}_+) := D_\varphi(\widetilde{U}_+, \widetilde{U}_-)$ is Q-graded.

Proposition 26. *In $D(\widetilde{U}_+)$, we have*

$$K_iK_j' = K_j'K_i, \quad K_i'E_j = q^{\alpha_j(H_i)}E_jK_i', \quad K_iF_j = q^{-\alpha_j(H_i)}F_jK_i, \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

Lemma 5. *The annihilator ideal $I_+ := I_{\widetilde{U}_+}$ of $\widetilde{U}_+$ is generated by the elements*

$$E_iE_j - E_jE_i, \quad |i - j| \geq 2,$$
$$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2, \quad |i - j| = 1.$$

Likewise, the annihilator ideal $I_- := I_{\widetilde{U}_-}$ of $\widetilde{U}_-$ is generated by the same elements with E_i replaced by F_i.

Let $U_+ := \widetilde{U}_+/I_+$ and $U_- := \widetilde{U}_-/I_-$. Then, φ induces a nondegenerate Hopf pairing between them, producing a Hopf algebra $D(U_+) := D_\varphi(U_+, U_-)$.

Definition 38. The Hopf algebra $U_q(\mathfrak{sl}_{N+1})$ is the quotient of $D(U_+)$ by the two-sided ideal generated by $K_i - K_i'$, $1 \leq i \leq N$.

That is, as an algebra, it is the $\mathbb{C}(q)$-algebra generated by $E_i, F_i, K_i^{\pm 1}$ $(1 \leq i \leq N)$ subject to relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$[K_i, K_j] = 0, \quad K_iE_j = q^{\alpha_j(H_i)}E_jK_i, \quad K_iF_j = q^{-\alpha_j(H_i)}F_jK_i, \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad 1 \leq i, j \leq N$$

$$[E_i, E_j] = 0, \quad [F_i, F_j] = 0, \quad \text{if } |i - j| \geq 2,$$

$$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0, \quad F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0, \quad \text{if } |i - j| = 1.$$

The universal R-matrix for $U_q(\mathfrak{sl}_{N+1})$ can be found e.g. in [Bur90]. Since $U_q(\mathfrak{sl}_{N+1})$ is infinite dimensional, the universal R-matrix is only formal, but it gives rise to an actual braiding for any finite dimensional representation of $U_q(\mathfrak{sl}_{N+1})$.

Example 14. $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}(q)$-algebra generated by $E, F, K^{\pm 1}$ subject to relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

It has the structure of a Hopf algebra given by[2]

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F,$$
$$\epsilon(K) = 1, \quad \epsilon(E) = 0, \quad \epsilon(F) = 0,$$
$$S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF.$$

It has a universal R-matrix

$$R = q^{\frac{1}{2}H\otimes H} \sum_{n\geq 0} q^{\frac{n(n-1)}{2}} \frac{(q-q^{-1})^n}{[n]!} (E^n \otimes F^n)$$

and a ribbon element

$$\theta = K^{-1} \sum S(R_{(2)}) R_{(1)},$$

where $R = \sum R_{(1)} \otimes R_{(2)}$.

12.1.2. *Specializations.* Let $A = \mathbb{C}[q, q^{-1}] \subset \mathbb{C}(q)$. Let U_A be the Hopf A-subalgebra of $U_q(\mathfrak{sl}_{N+1})$ generated by the elements

$$E_i, F_i, K_i^{\pm 1}, [K_i; 0], \quad 1 \leq i \leq N,$$

where

$$[K_i; 0] := \frac{K_i - K_i^{-1}}{q - q^{-1}}.$$

For $\epsilon \in \mathbb{C}^*$, the specialization $U_\epsilon := U_A/(q-\epsilon)U_A$ is a Hopf algebra over $\mathbb{C}$.

Proposition 27. *The algebra U_1 is generated by E_i, F_i, K_i, H_i $(1 \leq i \leq N)$, subject to the condition that all elements K_i are central and*

$$[H_i, E_j] = \alpha_j(H_i) K_i E_j, \quad [H_i, F_j] = -\alpha_j(H_i) K_i F_j,$$
$$[E_i, F_j] = \delta_{ij} H_i, \quad K_i^2 = 1,$$
$$[E_i, E_j] = 0, \quad [F_i, F_j] = 0, \quad \textit{if } |i-j| \geq 2,$$
$$[E_i, [E_i, E_j]] = 0, \quad [F_i, [F_i, F_j]] = 0, \quad \textit{if } |i-j| = 1.$$

In particular, $U_1/\langle K_i - 1 \mid 1 \leq i \leq N\rangle$ is isomorphic to $U(\mathfrak{sl}_{N+1})$.

Let $\epsilon \in \mathbb{C}^*$ be such that $\epsilon \neq 1$. Then the presentation of U_ϵ by generators and relations can be obtained from that of $U_q(\mathfrak{sl}_{N+1})$ by putting $q = \epsilon$. When ϵ is not a root of unity, then the Hopf pairing is still non-degenerate.

However, when ϵ is a primitive ℓ-th root of unity, then $E_\alpha^\ell, F_\alpha^\ell, K_i^\ell - 1$ $(\alpha \in \Phi^+,\ 1 \leq i \leq N)$ are in the kernel of the Hopf pairing and are in fact in the center of U_ϵ. Let u_ϵ (the *small quantum group*) be the quotient of U_ϵ by the two-sided ideal generated by those elements. Then, u_ϵ is a finite dimensional Hopf algebra.

Theorem 20. *If ℓ is odd and prime to $N+1$, then u_ϵ is braided; see e.g.* [KRT97, Theorem 4.4] *for the formula for the universal R-matrix.*

[2]We are following the convention of [Hab02], which is slightly different from that of [KRT97] which we have been using above; their coproducts are opposite to each other (and hence their antipodes are inverses of each other).

12.2. **Modular categories.** So far, we have been studying braided and ribbon categories and how to construct link invariants out of them. When it comes to 3-manifold invariants, it turns out that the appropriate categorical notion we want is that of a modular tensor category.

In this section, we mostly follow [Tur94, Ch. 2]; see also [BK01, Ch. 3].

Definition 39. A category is said to be an *Ab-category* if for any pair of its objects V, W, the set of morphisms $V \to W$ is an additive abelian group and the compositions are bilinear.

For monoidal Ab-categories, we also assume that the tensor product of morphism is bilinear.

Let $\mathcal{V}$ be a monoidal Ab-category. Then, $R = R_{\mathcal{V}} := \mathrm{End}(\mathbb{1})$ is a commutative ring with unit $\mathrm{id}_{\mathbb{1}}$ and is called the *ground ring* of $\mathcal{V}$. The abelian groups $\mathrm{Hom}(V, W)$ acquire the left R-module structure by $kf = k \otimes f$ where $k \in R$ and $f \in \mathrm{Hom}(V, W)$. The composition of morphisms is R-bilinear.

Definition 40. Let $\mathcal{V}$ be a ribbon Ab-category. An object V of $\mathcal{V}$ is said to be *simple* if

$$R \to \mathrm{End}(V)$$
$$r \mapsto r \otimes \mathrm{id}_V$$

is a bijection. That is, if $\mathrm{End}(V)$ is a free R-module of rank 1, generated by id_V.

For example, the tensor unit $\mathbb{1}$ is simple. An object isomorphic to or dual to a simple object is also simple.

Definition 41. Let $\{V_i\}_{i \in I}$ be a family of objects in a ribbon Ab-category $\mathcal{V}$. We say that an object V of $\mathcal{V}$ is *dominated* by the family $\{V_i\}_{i \in I}$ if there exists a finite set $\{V_{i(r)}\}_r$ of objects of this family (possibly with repetition) and a family of morphisms

$$\{f_r : V_{i(r)} \to V,\ g_r : V \to V_{i(r)}\}$$

such that

$$\mathrm{id}_V = \sum_r f_r g_r.$$

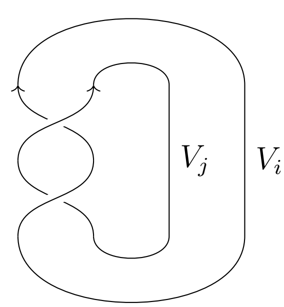
In other words, V is dominated by $\{V_{i(r)}\}_r$ if the image of the pairings

$$\{(g, f) \mapsto fg : \mathrm{Hom}(V, V_i) \otimes_R \mathrm{Hom}(V_i, V) \to \mathrm{End}(V)\}_{i \in I}$$

additively generate $\mathrm{End}(V)$.

For example, if the category $\mathcal{V}$ admits direct sums, then V is dominated by $\{V_{i(r)}\}_r$ if and only if, for some object W of $\mathcal{V}$, the direct sum $V \otimes W$ splits as a direct sum of a finite number of objects from this family.

Set $S_{i,j} := \mathrm{Tr}\big(\beta_{V_j,V_i} \circ \beta_{V_i,V_j}\big) \in R$ for each $i, j \in I$. That is, it is the invariant associated to the Hopf link colored by V_i and V_j:



.

This is a symmetric $I \times I$ matrix. Note, $S_{0,i} = S_{i,0} = \dim(V_i)$.

Definition 42. A *modular category* is a pair $(\mathcal{V}, \{V_i\}_{i\in I})$ consisting of a ribbon Ab-category $\mathcal{V}$ and a finite family of simple objects $\{V_i\}_{i\in I}$ of $\mathcal{V}$ satisfying the following properties:

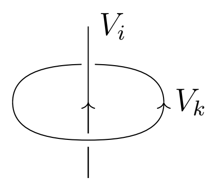
(1) (Normalization). There exists $0 \in I$ such that $V_0 = \mathbb{1}$.
(2) (Duality). For any $i \in I$, there is $i^* \in I$ such that the object V_{i^*} is isomorphic to $(V_i)^*$.
(3) (Domination). All objects of $\mathcal{V}$ are dominated by the family $\{V_i\}_{i\in I}$.
(4) (Non-degeneracy). The square matrix $S = (S_{i,j})_{i,j\in I}$ is invertible over R; that is, $\det(S)$ is invertible in R.

It is easy to see that S_{ij} is divisible by both $\dim(V_i)$ and $\dim(V_j)$. Non-degeneracy axiom implies that $\dim(V_i)$ is invertible in R. Non-degeneracy axiom also implies that the objects V_i, V_j with distinct i, j are not isomorphic.

A modular category is called *strict* if the underlying monoidal category is strict. MacLane's coherence theorem works in the setting of modular cagtegory as well, so we may (and will) restrict our attention to strict modular categories.

Lemma 6 (Schur lemma)**.** *Let* $(\mathcal{V}, \{V_i\}_{i\in I})$ *be a modular category. Then, for any distinct* $i, j \in I$*, we have* $\mathrm{Hom}(V_i, V_j) = 0$.

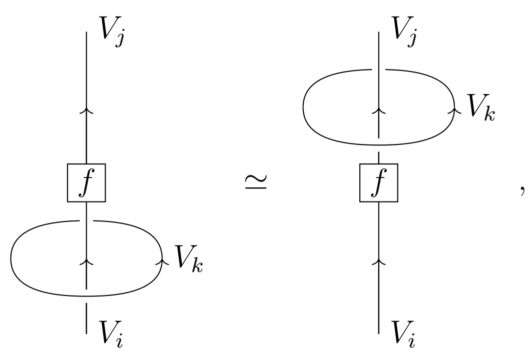
Proof. Firstly, note that, by simiplicity of the objects V_i's, the colored ribbon graph



must evaluate to some constant times id_{V_i}. By taking the trace, it is easy to see that this constant must be

$$\frac{S_{ik}}{\dim(V_i)} = \frac{S_{ik}}{S_{i0}}.$$

Let $f \in \mathrm{Hom}(V_i, V_j)$. The following two colored ribbon graphs are isotopic:



so they must evaluate to the same morphism. The left picture evaluates to $\frac{S_{ik}}{S_{i0}} f$ and the right picture evaluates to $\frac{S_{jk}}{S_{j0}} f$. Therefore, we should have

$$\left(\frac{S_{ik}}{S_{i0}} - \frac{S_{jk}}{S_{j0}}\right) f = 0,$$

for all $k \in I$. By multiplying $(S^{-1})_{ki}$ to the right and summing over $k \in I$, we get

$$\left(\frac{1}{S_{i0}} - \frac{\delta_{ij}}{S_{j0}}\right) f = 0.$$

In particular, if $i \neq j$, then f must be 0. □

13. Lecture 13 (Tue Mar 5, 2024)

13.1. Witten-Reshetikhin-Turaev invariants.

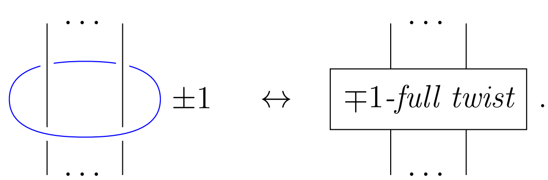
13.1.1. *Lightening review of 3d Kirby calculus.* Let $L \subset S^3$ be a framed link. A *Dehn surgery* along L is a process of gluing solid tori to the boundary of the link complement in the following way. For each component L_i of L, let μ_i and λ_i be the meridian and longitude on the torus boundary of a tubular neighborhood of L_i. Note, μ_i is independent of the framing of L_i while λ_i is dependent on the framing. We glue the boundary of a solid torus along the torus boundary in such a way that it kills the homology cycle represented by the longitude λ_i. In the end, we get a closed oriented 3-manifold $Y_L = S^3(L)$.

Equivalently, one can glue 2-handles on B^4 along L_i's, with appropriate framing. Let's call the resulting 4-manifold W_L. The Dehn-surgered 3-manifold $S^3(L)$ is ∂W_L.

The signature of W_L (i.e. the signature of the intersection form on $H_2(W_L; \mathbb{R})$) is denoted by $\sigma(L)$. Equivalently, it is the signature of the linking matrix of L.

Theorem 21 (Lickorish–Wallace)**.** *Every connected closed oriented 3-manifold can be obtained from S^3 by performing a Dehn surgery along some framed link $L \subset S^3$.*

Theorem 22 (Kirby, Fenn-Rourke)**.** *If two framed links $L_1, L_2 \subset S^3$ give rise to the same 3-manifold via Dehn surgery, then they are related by a finite sequence of the following moves:*



This move is called the *Fenn-Rourke move*, which simplifies Kirby's original formulation, which involves two kinds of moves – "blow-ups" and "handle slides". We will simply call a Fenn-Rourke move involving a ±1-framed unknot component a *±1-move.* A −1-move with no other strands linked to the −1-framed unknot will be called a *special −1-move.* It is known that we may restrict to +1-moves and special −1-moves.

Therefore, in order to create an invariant of closed oriented 3-manifolds, it suffices to produce an invariant of framed links which is invariant under +1-moves and special −1-moves.

13.1.2. *WRT invariants.* Let $(\mathcal{V}, \{V_i\}_{i\in I})$ be a modular category with ground ring R.

Given a framed oriented link $L \subset S^3$ with m components, let $\mathrm{col}(L)$ be the set of functions from the set of components of L to I. For each $c \in \mathrm{col}(L)$, let L_c be the corresponding colored link. Define

$$\langle L \rangle := \sum_{c \in \mathrm{col}(L)} \dim(c)\, F(L_c) \ \in R,$$

where

$$\dim(c) := \prod_{1 \leq n \leq m} \dim(V_{c(L_n)}),$$

and F is the Reshetikhin-Turaev functor (Theorem 9). More generally, given a $\mathcal{V}$-colored framed oriented link K in the complement of L, define

$$\langle L, K\rangle := \sum_{c\in\mathrm{col}(L)} \dim(c)\, F(L_c \cup K) \ \in R.$$

Lemma 7. *$\langle L\rangle$ and $\langle L, K\rangle$ does not depend on the orientation of L.*

Proof. This is easy to see from the duality axiom of a modular category. □

As we will see, $\langle L, K\rangle$ almost defines an invariant of the pair (Y_L, K), in a sense that, under the Fenn-Rourke move, it only gets multiplied by a simple factor. To describe this simple factor, we need two elements Δ and D of R:

Definition 43. Set

$$\Delta := \langle O_{-1}\rangle = \sum_{i\in I} v_i^{-1}(\dim(V_i))^2 \ \in R,$$

where O_{-1} denotes the -1-framed unknot, and $v_i \in R$ is the invertible element determined by $\theta_{V_i} = v_i\, \mathrm{id}_{V_i}$.

Definition 44. A *rank* of $\mathcal{V}$ is an element $D \in R$ such that

$$D^2 = \langle O_0\rangle = \sum_{i\in I}(\dim(V_i))^2,$$

where O_0 denotes the 0-framed unknot.

Note, a rank may not exists and even if it exists, it may not be unique. If it doesn't exist, we may always formally add such an element to R by extending the ground ring to $\widetilde{R} := R[x]/(x^2 - \sum_{i\in I}(\dim(V_i))^2)$.

Lemma 8. *Δ and any rank D of $\mathcal{V}$ are invertible in R.*

We will prove this lemma in the proof of the following theorem, which is the main theorem of this section:

Theorem 23. *Let $(\mathcal{V}, \{V_i\}_{i\in I})$ be a modular category with a rank $D \in R$. Let Y be a 3-manifold obtained as a surgery on a framed link $L \subset S^3$ with m components. Then*

$$\tau(Y) := \Delta^{\sigma(L)} D^{-\sigma(L)-m-1}\langle L\rangle \ \in R$$

is a topological invariant of Y.

More generally, if K is a $\mathcal{V}$-colored framed oriented link in Y, represented by a colored framed oriented link $K \subset S^3 \setminus L$, then

$$\tau(Y, K) := \Delta^{\sigma(L)} D^{-\sigma(L)-m-1}\langle L, K\rangle \ \in R$$

is a topological invariant of the pair (Y, K).

This invariant is called the *Witten-Reshetikhin-Turaev (WRT) invariant.*

Example 15. Here are some simple example computations:

- Taking $L = \emptyset$, we get $\tau(S^3) = D^{-1}$. More generally, $\tau(S^3, K) = D^{-1}F(K)$.
- Taking $L = O_0$, we get $\tau(S^1 \times S^2) = 1$.

- Taking $L = O_n$, we get

$$\tau(L(n,1)) = \Delta^{\mathrm{sgn}(n)} D^{-\mathrm{sgn}(n)-2} \sum_{i\in I} v_i^n (\dim(V_i))^2.$$

Note, by setting $n = 1$, in which case we should get S^3, we get the following identity:

$$D^{-1} = \langle O_{-1}\rangle D^{-3} \langle O_1\rangle,$$

or in other words,

$$\langle O_0\rangle = \langle O_1\rangle\langle O_{-1}\rangle.$$

- If L_1 and L_2 are unlinked, then $Y_{L_1\cup L_2} = Y_{L_1}\# Y_{L_2}$. It follows that

$$\tau(Y_{L_1}\# Y_{L_2}) = D\tau(Y_{L_1})\tau(Y_{L_2}) = \frac{\tau(Y_{L_1})\tau(Y_{L_2})}{\tau(S^3)}.$$

Remark 16. The product $D^{b_1(M)+1}\tau(Y,K)$ involves only even power of D, so it is independent of the choice of D. One can get rid of the dependence of D in this way if one wishes, but the normalization used above is a more natural one, from the 3d TQFT point of view.

Proof of Theorem 23. We reduce the theorem into Lemma 9 and Lemma 10.

Lemma 9. *Let $\{d_i\}_{i\in I} \in R^I$ be the unique solution to the following system of equations:*

$$\sum_{i\in I} d_i v_i v_j S_{i,j} = \dim(V_j).$$

Set

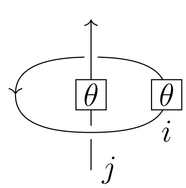
$$\langle L, K\rangle' := \sum_{c\in \mathrm{col}(L)} \prod_{1\le n\le m} d_{c(L_n)} F(L_c \cup K) \ \in R.$$

Then, the element $\langle L, K\rangle' \in R$ does not depend on the choice of orientation of L, and this element is invariant under $+1$-moves on (L, K).

Proof of Lemma 9. It is easy to see that $S_{ij} = S_{i^*j^*}$ and $v_{i^*} = v_i$. It follows that $d_i = d_{i^*}$. This implies that $\langle L, K\rangle' \in R$ is independent of the choice of orientation of L.

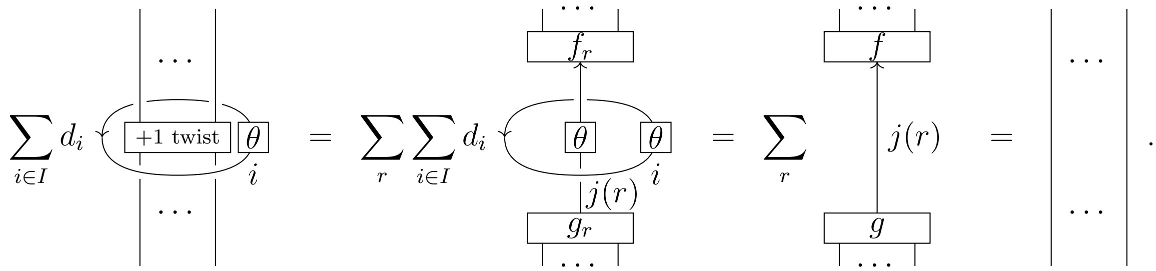
For invariance of $\langle L, K\rangle'$ under $+1$ moves, we observe that, by the definition of d_i's,

$$\sum_{i\in I} d_i$$



$$= \sum_{i\in I} d_i v_i v_j \frac{S_{ij}}{\dim(V_j)} \cdot \mathrm{id}_{V_j} = \mathrm{id}_{V_j}$$

We need to show the same for arbitrary number of strands linking the unknot component. The case with 0 strand follows from the same identity, specialized to $j = 0$. In case of one or more strands, using the axiom of domination, we can decompose the identity morphism on any object W of $\mathcal{V}$ into morphisms that factor through V_i's:



$\square$

To get a topological invariant of 3-manifolds, it only remains to check how $\langle L, K\rangle'$ behave under the special -1-move. Under the special -1-move, we have

$$\langle L_{-1}, K\rangle' = \langle L, K\rangle' \sum_{i\in I} d_i v_i^{-1} \dim(V_i).$$

Therefore, if $\sum_{i\in I} d_i v_i^{-1} \dim(V_i) \in R$ is invertible, then we will be able to make this a topological invariant, by multiplying an appropriate power of this number.

14. Lecture 14 (Thu Mar 7, 2024)

14.1. **Witten-Reshetikhin-Turaev invariants (cont.)**

Proof of Theorem 23 (cont.)

Lemma 10. *Set*

$$x := \langle O_0\rangle' = \sum_{i\in I} d_i \dim(V_i) \ \in R.$$

(1) *For any $i \in I$, we have*

$$\sum_{j\in I} d_j S_{j,i} = x\delta_{i,0}.$$

(2) *For any $i, j \in I$, we have*

$$\sum_{k\in I} \frac{d_k}{\dim(V_k)} S_{i,k} S_{k,j} = x\delta_{i^* j}.$$

(3) *For any $i \in I$, we have*

$$\sum_{j\in I} d_j v_j^{-1} v_i^{-1} S_{ji} = x d_i.$$

(4) *For any $i \in I$, we have*

$$d_i = d_0 \dim(V_i).$$

Here are some immediate consequences:

- The first identity (1) implies that x is a common divisor of $\{d_i\}_{i\in I}$, but from the definition of d_i's, it should be invertible.
- The second identity (2) implies that the matrix

$$\{x^{-1} \frac{d_i S_{i,j^*}}{\dim(V_i)}\}_{i,j\in I}$$

is the inverse of S. It also follows that all d_i's are invertible in R.

- The third identity (3) implies that
$$\sum_{i\in I} d_i v_i^{-1}\dim(V_i) = x d_0,$$
which the factor we obtain under the special -1-move, and we have just seen that this is invertible in R.

To this end, define
$$\tau'(Y_L, K) := (x d_0)^{-\sigma_-(L)}\langle L, K\rangle',$$
where $\sigma_-(L) = \frac{m-\dim H_1(Y;\mathbb{R})-\sigma(L)}{2}$ denotes the number of negative eigenvalues of the intersection form of W_L. It is easy to see that any $+1$-move preserves $\sigma_-(L)$, and any special -1-move adding a -1-framed unknot component increases $\sigma_-(L)$ by 1. Therefore, Lemma 9 and (1), (2), (3) of Lemma 10 implies that τ' is a topological invariant.

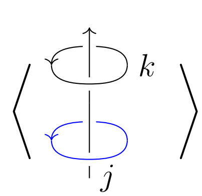
It remains to relate τ' with τ.

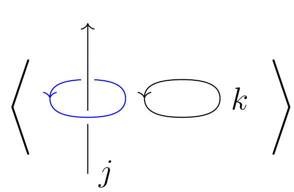
- Plugging the identity (4) into $\sum_{i\in I} d_i v_i^{-1}\dim(V_i) = x d_0$, we get
$$x = \sum_{i\in I} v_i^{-1}(\dim(V_i))^2 =: \Delta.$$
Therefore Δ is invertible.
- Moreover,
$$\Delta = x := \sum_{i\in I} d_i \dim(V_i) = d_0\sum_{i\in I}(\dim(V_i))^2 = d_0 D^2.$$
Therefore, D is invertible with $D^{-1} = d_0 D\Delta^{-1}$.

That is, Lemma 10 imply Lemma 8.

Finally, replacing $d_i = d_0\dim(V_i)$, $x = \Delta$ and $d_0 = \Delta D^{-2}$ in the definition of τ', we get
$$\tau'(Y, K) = \Delta^{b_1(Y)} D^{-b_1(Y)+1}\tau(Y, K).$$
Hence, topological invariance of τ' implies that of τ. □

Proof of Lemma 10. Proof of (1): We have[3]

$$\sum_{i\in I} d_i S_{i,j}\frac{S_{k,j}}{\dim(V_j)^2}\mathrm{id}_{V_j} =$$


$$\xlongequal[\text{See Fig. 3.6 in Turaev}]{\text{a sequence of } +1 \text{ moves}}$$


$$= \sum_{i\in I} d_i S_{i,j}\frac{\dim(V_k)}{\dim(V_j)}\mathrm{id}_{V_j},$$

which implies
$$\left(\sum_{i\in I} d_i S_{i,j}\right) = \left(\sum_{i\in I} d_i S_{i,j}\right)\frac{S_{k,j}}{\dim(V_j)\dim(V_k)}.$$

[3]See Fig. 3.6 in [Tur94] for the sequence of $+1$ moves realizing the handle slide along the blue 0-framed unknot. This is the most technical part of the proof.

Nondegeneracy of S and invertibility of V_j, V_k implies that the matrix

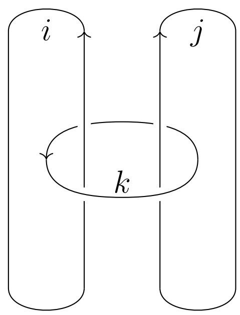
$$S' := \left(\frac{S_{j,k}}{\dim(V_j)\dim(V_k)}\right)_{j,k}$$

is nondegenerate. Note, $S'_{0,k} = 1$ for all $k \in I$, so for any $j \neq 0$, $S'_{j,k}$ can't be 1 for all $k \in I$. Therefore,

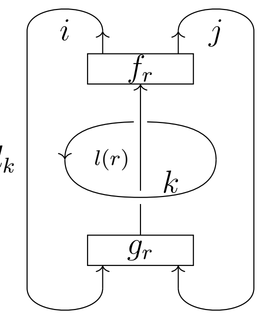
$$\sum_{i\in I} d_i S_{i,j} = \delta_{j,0} \sum_{i\in I} d_i \dim(V_i) = \delta_{j,0}\, x.$$

Proof of (2): Using domination and (1), we have

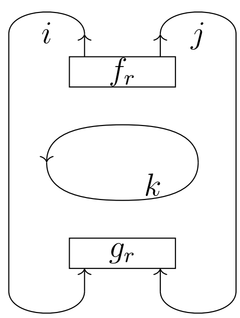
$$\sum_{k\in I} \frac{d_k S_{i,k} S_{j,k}}{\dim(V_k)} = \sum_{k\in I} d_k$$



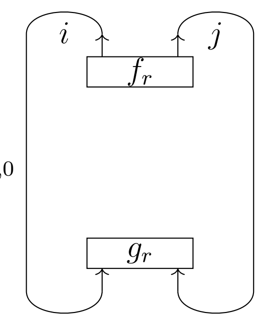
$$= \sum_r \sum_{k\in I} d_k$$



$$= \sum_r \delta_{l(r),0} \sum_{k\in I} d_k$$



$$= x \sum_r \delta_{l(r),0}$$

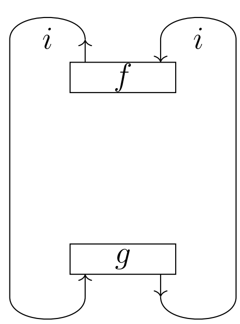


.

From Schur lemma (Lemma 6), it is easy to see that

$$\mathrm{Hom}(\mathbb{1}, V_i \otimes V_j) = 0 = \mathrm{Hom}(V_i \otimes V_j, \mathbb{1})$$

unless $j = i^*$, in which case $\mathrm{Hom}(\mathbb{1}, V_i \otimes V_i^*)$ is a free rank 1 R-module spanned by $\overleftarrow{\cup}_{V_i}$, and $\mathrm{Hom}(V_i \otimes V_i^*, \mathbb{1})$ is a free rank 1 R-module spanned by $\overrightarrow{\cap}_{V_i}$. Therefore, without loss of generality, we may assume that there is only one r for which $l(r) = 0$. It remains to show that

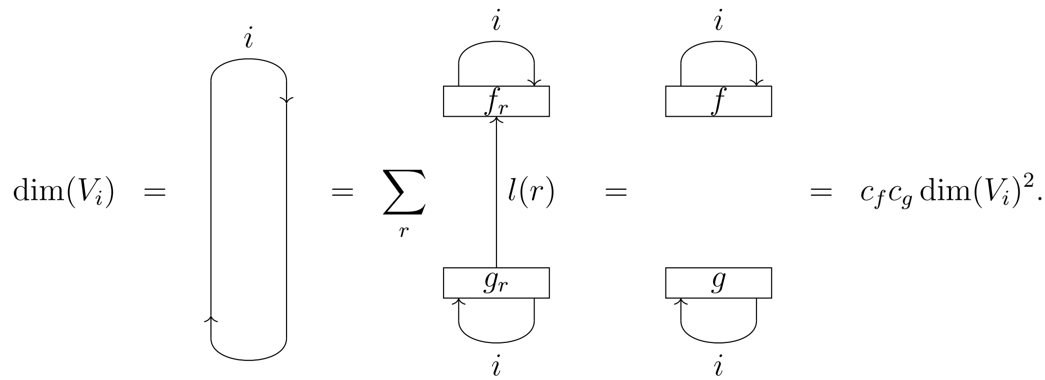


$$\stackrel{?}{=} 1.$$

Let $c_f, c_g \in R$ be constants determined by $f = c_f \overleftarrow{\cup}_{V_i}$ and $g = c_g \overrightarrow{\cap}_{V_i}$. Then, the LHS of the above picture evaluates to

$$c_f c_g \dim(V_i).$$

On the other hand, we have, using Schur lemma again,



Therefore, $c_f c_g \dim(V_i) = 1$.

Proof of (3): From the definition of d_i's,

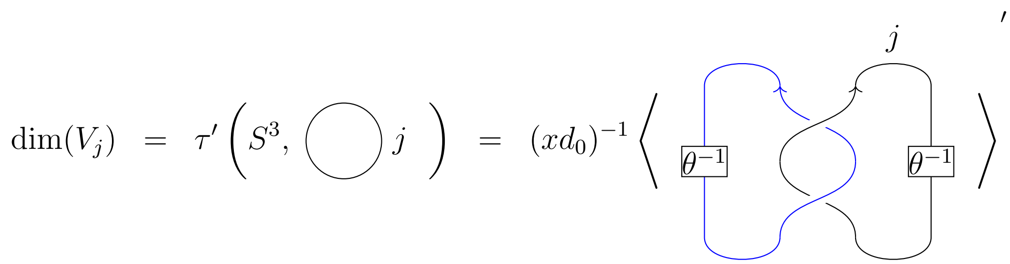
$$\sum_{i\in I} d_i v_i v_j S_{i,j} = \dim(V_j)$$

$$\Rightarrow \quad \sum_{i\in I} d_i d_j v_i v_k^{-1} \frac{S_{i,j}S_{j,k}}{\dim(V_j)} = d_j v_j^{-1} v_k^{-1} S_{j,k}$$

$$\overset{(2)}{\Rightarrow} \quad \sum_{i\in I} d_i v_i v_k^{-1} x \delta_{i^*,k} = \sum_{j\in I} d_j v_j^{-1} v_k^{-1} S_{j,k}$$

$$\Rightarrow \quad x d_k = \sum_{j\in I} d_j v_j^{-1} v_k^{-1} S_{j,k}.$$

Proof of (4): We have



$$= \quad (xd_0)^{-1} \sum_{i\in I} d_i v_i^{-1} v_j^{-1} S_{i,j} \quad \overset{(3)}{=} \quad (xd_0)^{-1} x d_j = \frac{d_j}{d_0}\ .$$

Therefore,

$$d_j = d_0 \dim(V_j).$$

□

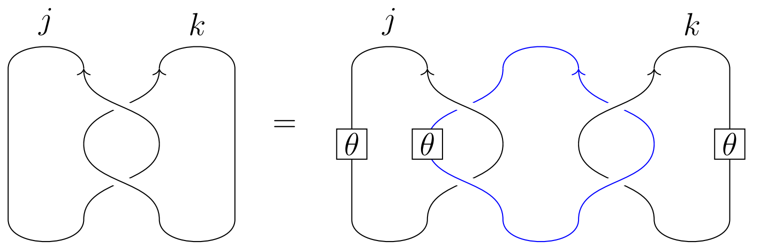
14.1.1. $\mathrm{SL}_2(\mathbb{Z})$-*action.* Consider the square matrices

$$S = (S_{i,j})_{i,j\in I}, \quad T := (\delta_{i,j} v_i)_{i,j\in I}, \quad J := (\delta_{i^*,j})_{i,j\in I}.$$

Both S and T commute with J. We have, from (2) and (4),

$$S^2 = d_0^{-1} x J = D^2 J.$$

Moreover, by computing τ for a colored Hopf link in S^3 in two different ways



we get

$$D^{-1}S_{j,k^*} = \Delta D^{-3} v_j v_k \sum_{i\in I} v_i S_{ji} S_{ik},$$

or equivalently,

$$SJ = \Delta D^{-2} TSTST.$$

The modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which satisfy relations

$$s^4 = 1, \quad (ts)^3 = s^2.$$

From our observations above, we obtain a projective representation

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) &\to \mathrm{PGL}_I(R) \\ s &\mapsto D^{-1}S, \\ t &\mapsto T^{-1}. \end{aligned}$$

Indeed,

$$(D^{-1}S)^4 = D^{-4}S^4 = D^{-4}(D^2J)^2 = 1,$$

and

$$(D^{-1}T^{-1}S)^3 = D^{-3}(T^{-1}ST^{-1})ST^{-1}S = \Delta D^{-5} JST(SS)T^{-1}S = \Delta D^{-3}S^2 = \Delta D^{-1}(D^{-1}S)^2.$$

It is because of the factor ΔD^{-1} we get a projective representation.

14.1.2. *Examples.*

Example 16. The ribbon category $\mathrm{Proj}(R)$ that we saw in Example 4 is modular. The set $\{V_i\}_{i\in I}$ consists of one element, R, and the matrix S is (1).

Example 17. The ribbon category $\mathcal{V}(G,R,c,\varphi)$ that we saw in Example 5, with the family $\{V_g\}_{g\in G}$ consisting of all objects of this category, is modular if and only if G is a finite group and the matrix $(c(g,h)c(h,g))_{g,h\in G}$ is invertible over R.

As a special case, consider the following. For a fixed positive odd integer k and a primitive k-th root of unity $\zeta = \zeta_k$, set

- $G = \mathbb{Z}/k\mathbb{Z}$,
- $R = \mathbb{Z}[\zeta]$ (or just $\mathbb{C}$),
- $c : G \times G \to R^*$, $(i,j) \mapsto \zeta^{ij}$,
- $\varphi : G \to R^*$, $i \mapsto 1$.

The matrix $S = (\zeta^{2ij})_{i,j\in\mathbb{Z}/k\mathbb{Z}}$ has determinant

$$\det\big(\zeta^{2ij}\big)_{i,j\in\mathbb{Z}/k\mathbb{Z}} = \prod_{0\le i<j\le k-1} (\zeta^{2j}-\zeta^{2i}),$$

and this is non-zero since we are assuming that k is odd. This is the modular category behind $U(1)$ *Chern-Simons theory*. The corresponding WRT invariants are Gauss sums and have particularly simple formulas.

Example 18. Let G be a finite group. We saw in Example 12 that the quantum double $D(G) := D(\Bbbk[G])$ of the group algebra $\Bbbk[G]$ is a ribbon Hopf algebra, and that the category $\operatorname{Rep} D(G)^{\mathrm{fin}}$ of finite dimensional representations of $D(G)$ is a ribbon category.

A representation V of $D(G)$ is the same as a representation of $\Bbbk[G]$ with G-grading $V = \bigoplus_{g\in G} V_g$ (where $V_g = \delta_g V$) satisfying $xV_g \subset V_{xgx^{-1}}$. Note that each V_g is a representation of $\Bbbk[Z(g)]$, and that for each $v \in V$, $\Bbbk[Z(g)]\delta_g v$ is an irreducible representation π of $\Bbbk[Z(g)]$. Moreover,

$$V_{\overline{g},\pi} := \Bbbk[G]\delta_g v = \bigoplus_{xgx^{-1}\in\overline{g}} x\pi$$

is an irreducible $D(G)$-module. Hence, for any $v \in V$, the submodule of V generated by v can be decomposed into irreducible modules as

$$D(G)v = \bigoplus_{\overline{g}\in\overline{G}} \Bbbk[G]\delta_g v.$$

Therefore, every finite dimensional representation of $D(G)$ can be decomposed into irreducible ones, i.e., $\operatorname{Rep} D(G)^{\mathrm{fin}}$ is a semisimple category.

$\operatorname{Rep} D(G)^{\mathrm{fin}}$ is in fact a modular category:

- Simple objects $V_{\overline{g},\pi}$ are labeled by a conjugacy class $\overline{g} \in \overline{G}$ and an isomorphism class $\pi \in \widehat{Z(g)}$ of irreducible representations of the centralizer $Z(g)$; $V_{\overline{g},\pi} := \bigoplus_{xgx^{-1}\in\overline{g}} x\pi$.
- Dual objects are given by $V^*_{\overline{g},\pi} \simeq V_{\overline{g^{-1}},\pi^*}$.
- The S and T matrices are given by

$$S_{(\overline{g},\pi),(\overline{g'},\pi')} = \frac{|G|}{|Z(g)||Z(g')|} \sum_{\substack{h\in G\\ hg'h^{-1}\in Z(g)}} \operatorname{Tr}_\pi(hg'^{-1}h^{-1}) \operatorname{Tr}_{\pi'}(h^{-1}g^{-1}h),$$

$$T_{(\overline{g},\pi),(\overline{g'},\pi')} = \delta_{(\overline{g},\pi),(\overline{g'},\pi')} \frac{\operatorname{Tr}_\pi(g)}{\operatorname{Tr}_\pi(e)}.$$

- $\Delta = |G|$ and $D = \pm|G|$.

See [BK01, Theorem 3.2.1] for a proof.

Exercise 9. Compute the WRT invariant for lens spaces, in the examples above.

Example 19. Consider the quantum group $U_q(\mathfrak{g})$ at $q = e^{\frac{\pi i}{mk}}$, where $k \in \mathbb{Z}_+$ and $m := \frac{\langle\alpha,\alpha\rangle}{2}$ for a long root α ($m = 1$ if $\mathfrak{g}$ is simply-laced). Let $\mathcal{C}(\mathfrak{g},k)$ be the category of finite dimensional representations of $U_q(\mathfrak{g})$ over $\mathbb{C}$ with weight decomposition:

$$V = \bigoplus_{\lambda\in P} V^\lambda, \quad q^H|_{V^\lambda} = q^{(H,\lambda)}\mathrm{id}_{V^\lambda},$$

$$E_i^{(n)}(V^\lambda) \subset V^{\lambda+n\alpha_i}, \quad F_i^{(n)}(V^\lambda) \subset V^{\lambda-n\alpha_i}.$$

It is known that $\mathcal{C}(\mathfrak{g}, k)$ is a ribbon category. It contains, as a full subcategory, the category $\mathcal{T}$ of tilting modules, which is also a ribbon category.

Let $\mathcal{C}^{\text{int}}(\mathfrak{g}, k)$ $(k \geq h^\vee)$ be the category of tilting modules where we quotient out *negligible morphisms*, i.e. morphisms $f : T_1 \to T_2$ for which $\text{Tr}_q(fg) = 0$ for all $g : T_2 \to T_1$. Then $\mathcal{C}^{\text{int}}(\mathfrak{g}, k)$ is a (semisimple, abelian) modular category whose simple objects are the Weyl modules

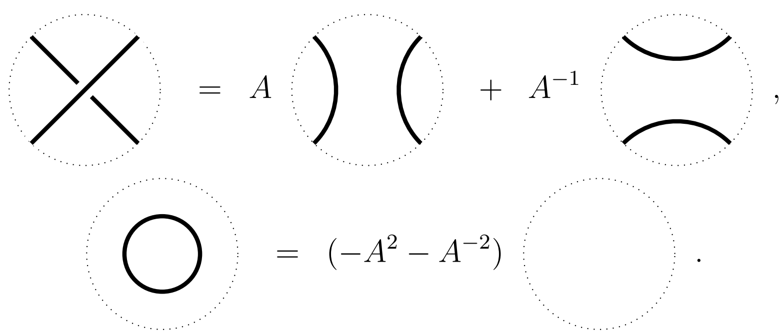
$$V_\lambda, \text{ with } \lambda \in P_+, \ (\lambda + \rho, \theta^\vee) < k.$$

See [BK01, Ch. 3.3] for details. This is the modular category behind Chern-Simons 3d TQFT.

15. Lecture 15 (Tue Mar 26, 2024)

15.1. **Skein modules, algebras, and categories.** Let's start from the basic definition of skein modules.

Definition 45 ([Prz91, Tur88]). The $\mathfrak{sl}_2$ *(or Kauffman bracket) skein module* $\text{Sk}_A^{\mathfrak{sl}_2}(Y)$ of a 3-manifold Y is the $\mathbb{Z}[A^{\pm 1}]$-module freely spanned by isotopy classes of framed, unoriented links in Y, modulo the following skein relations (all in blackboard framing):



In particular, the well-definedness of Kauffman bracket implies that

$$\text{Sk}_A^{\mathfrak{sl}_2}(\mathbb{R}^3) \cong \mathbb{Z}[A^{\pm 1}]$$
$$[K] \mapsto \langle K \rangle.$$

When Y is of the form $\Sigma \times \mathbb{R}$, the skein module is naturally an algebra, with the algebra structure given by stacking along the $\mathbb{R}$-direction. In this case, we call it the $\mathfrak{sl}_2$ *skein algebra* of Σ and denote it by $\text{SkAlg}_A^{\mathfrak{sl}_2}(\Sigma)$.

Even better, we can think of it in terms of an algebroid (i.e. a linear category):

Definition 46. The $\mathfrak{sl}_2$ *skein category* $\text{SkCat}_A^{\mathfrak{sl}_2}(\Sigma)$ associated to an oriented surface Σ consists of the following data:

(1) An object of $\text{SkCat}_A^{\mathfrak{sl}_2}(\Sigma)$ is a finite set of framed points on Σ.
(2) The morphism space between x and y is the skein module of $\Sigma \times I$ with boundary condition x (resp. y) on $\Sigma \times \{0\}$ (resp. $\Sigma \times \{1\}$). That is, a morphism between x and y is represented by a framed tangle in $\Sigma \times I$ interpolating x and y.
(3) Composition of two morphisms are defined in an obvious way, by stacking.

Remark 17. The notion of skein modules and skein categories can be generalized to any ribbon category, using the Reshetikhin-Turaev functor we studied in Section ??. That is, given a $\mathbb{k}$-linear ribbon category $\mathcal{V}$, the *skein module* $\text{Sk}_{\mathcal{V}}(Y)$ is the $\mathbb{k}$-vector space freely spanned by $\mathcal{V}$-colored ribbon graphs in Y, modulo skein relations, which are all the $\mathcal{V}$-colored ribbon graphs in $D^2 \times I$ which evaluate to 0 under Reshetikhin-Turaev functor.

The skein category $\mathrm{SkCat}_{\mathcal{V}}(\Sigma)$ can be defined in a similar way.

15.2. **Classical limit and character varieties.** One of the most important properties of skein modules is that they quantize the character variety; [Bul97, PS00].

For concreteness, let's focus on the $\mathfrak{sl}_2$ skein module. When $A = \pm 1$, it is clear from the skein relations that the skein module is no longer sensitive to the crossings; that is, $\mathrm{Sk}^{\mathfrak{sl}_2}_{A=\pm 1}(Y)$ is a commutative algebra, with the algebra structure given by superposing two links.

Definition 47. Let $R(Y)$ be the $\mathrm{SL}_2(\mathbb{C})$-*representation variety* of $\pi_1(Y)$; that is,

$$R(Y) := \{\rho : \pi_1(Y) \to \mathrm{SL}_2(\mathbb{C})\}.$$

For each $\rho : \pi_1(Y) \to \mathrm{SL}_2(\mathbb{C})$, let $\chi_\rho := \mathrm{Tr}\,\rho : \pi_1(Y) \to \mathbb{C}$ be the associated character. The $\mathrm{SL}_2(\mathbb{C})$-*character variety* $X(Y)$ is defined as

$$X(Y) := \{\chi_\rho : \overline{\pi_1(Y)} \to \mathbb{C} \mid \rho \in R(Y)\}.$$

For any conjugacy class $\overline{\gamma} \in \overline{\pi_1(Y)}$, let $t_{\overline{\gamma}} \in \mathbb{C}[X(Y)]$ be the function on $X(Y)$ defined by $t_{\overline{\gamma}}(\chi) := \chi(\overline{\gamma})$. Let T be the ring of functions on $X(Y)$ generated by $t_{\overline{\gamma}}$'s.

Lemma 11. *For any $x, y \in \mathrm{SL}_2(\mathbb{C})$,*

$$\mathrm{Tr}(xy) + \mathrm{Tr}\big(xy^{-1}\big) = \mathrm{Tr}(x)\,\mathrm{Tr}(y).$$

Proof. This follows easily from the identity $y + y^{-1} = \mathrm{Tr}(y)I$. □

Using this lemma and finite-generation of $\pi_1(Y)$, one can show that:

Proposition 28 ([CS83])**.** *T is finitely generated.*

Let $t_{\overline{\gamma_1}}, \cdots, t_{\overline{\gamma_m}}$ be generators of T. Then, $t := (t_{\overline{\gamma_1}}, \cdots, t_{\overline{\gamma_m}})$ defines an injective map

$$t : X(Y) \to \mathbb{C}^m.$$

Theorem 24 ([CS83])**.** *$t(X(Y)) \subset \mathbb{C}^m$ is a closed algebraic subset. Therefore, the character variety $X(Y)$ is an affine variety.*

The following theorems give a precise relationship between the $\mathrm{SL}_2(\mathbb{C})$-character variety and $\mathrm{Sk}^{\mathfrak{sl}_2}_{A=-1}(Y)$.

Theorem 25 ([Bul97])**.** *The map*

$$\Phi : \mathrm{Sk}^{\mathfrak{sl}_2}_{A=-1}(Y) \to \mathbb{C}[X(Y)]$$
$$[K] \mapsto \Phi(K) : \chi \mapsto -\chi(K)$$

is a well-defined surjective algebra homomorphism.

Proof. We need to check that this map respects the two skein relations. For the first one,

$\Phi\Big($  $+$  $+$  $\Big)(\chi) \;=\; -\mathrm{Tr}(xy) - \mathrm{Tr}\big(xy^{-1}\big) + \mathrm{Tr}(x)\,\mathrm{Tr}(y) = 0,$

and for the second one,

$\Phi($  $+ 2) \;=\; -\mathrm{Tr}(I) + 2 = 0.$

□

Theorem 26 ([Bul97])**.** *The kernel of* Φ *is the nilradical (i.e. the ideal consisting of nilpotent elements) of* $\mathrm{Sk}^{\mathfrak{sl}_2}_{A=-1}(Y)$*.*

A better way to formulate the relationship between the skein module and character variety is to enhance the character variety into a (possibly non-reduced) scheme.

Definition 48. The *character scheme* $\mathcal{X}(Y) = \mathrm{Hom}(\pi_1(Y), \mathrm{SL}_2(\mathbb{C})) /\!\!/ \mathrm{SL}_2(\mathbb{C})$ is defined to be the GIT quotient

$$\mathcal{X}(Y) = \mathrm{Spec}(\mathbb{C}[R(Y)]^{\mathrm{SL}_2(\mathbb{C})}).$$

The character variety $X(Y)$ is then the algebraic set underlying $\mathcal{X}(Y)$. That is,

$$\mathbb{C}[X(Y)] = \mathbb{C}[\mathcal{X}(Y)]/\sqrt{0}.$$

Theorem 27 ([PS00])**.** *There is a natural* $\mathbb{C}$*-algebra isomorphism* $\mathrm{Sk}^{\mathfrak{sl}_2}_{A=-1}(Y) \cong \mathbb{C}[\mathcal{X}(Y)]$*.*

16. Lecture 16 (Tue Apr 2, 2024)

16.1. Quantum Teichmüller spaces. [To be written]

16.2. Quantum trace map. [BW11] [To be written]

Theorem 28 ([BW11])**.** *There is an algebra embedding*

$$\mathrm{Tr}^{\omega}_{\lambda} : \mathrm{SkAlg}^{\mathfrak{sl}_2}_{A=\omega^{-2}}(\Sigma) \to \mathcal{Z}^{\omega}_{\lambda}$$

from the Kauffman bracket skein algebra into the Checkhov-Fock square root algebra.

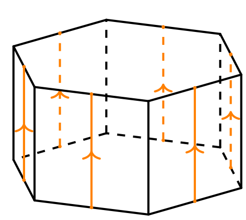
17. Lecture 17 (Thu Apr 4, 2024)

17.1. **Stated skein modules.** It is possible to extend the notion of skein modules to allow the skeins to end on a boundary marking; [BW11, Lê18, CL22a, CL22b, PP24].

17.1.1. *Stated skein algebras of surfaces.* We mostly follow [CL22a] in this subsection.

By a *bordered, punctured surface*, we mean an oriented surface possibly with boundary and punctures, such that each connected component of the boundary is an interval.

Let Σ be a bordered, punctured surface. For each boundary interval, fix a base point; let P be the set of those base points. Then, $P \times I \subset \Sigma \times I$ consists of oriented intervals (orientation is induced from that of I); see the figure below, in case Σ is a hexagon D_6:

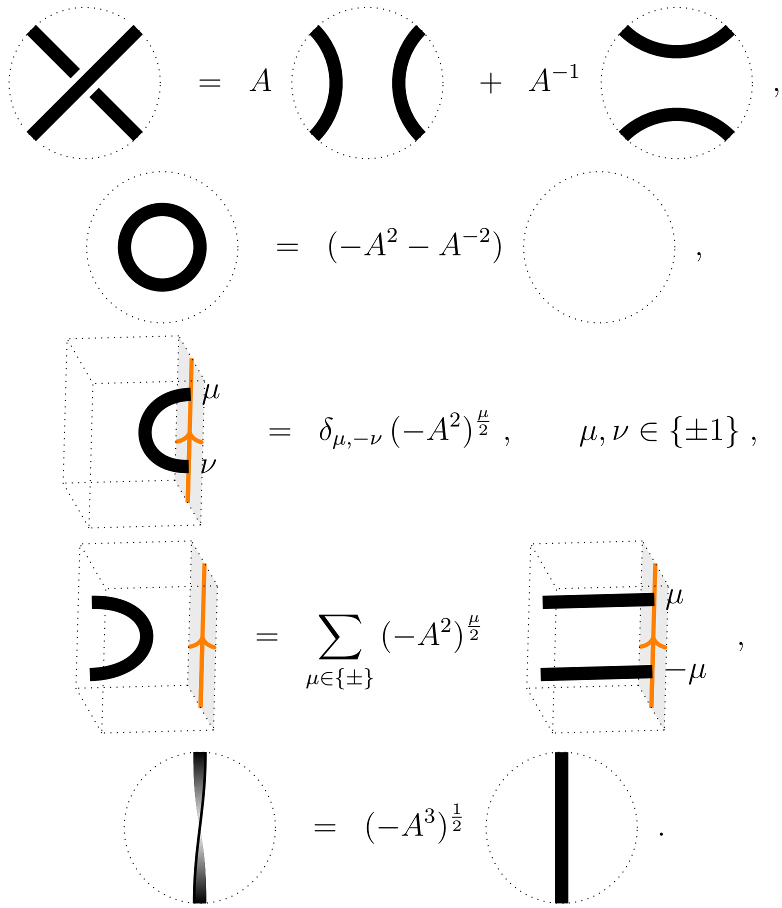


We call $P \times I$ the *boundary marking.*

Let $R := \mathbb{Z}[A^{\pm\frac{1}{2}}, (-A^2)^{\pm\frac{1}{2}}]$.

Definition 49. The *stated* $\mathfrak{sl}_2$*-skein algebra* $\mathrm{SkAlg}^{\mathfrak{sl}_2}_A(\Sigma, P)$ is the free R-module spanned by (2-framed) ribbon tangles in $\Sigma \times I$ with boundaries lying flat on the boundary marking

$P \times I$, with each boundary points equipped with a sign $\in \{\pm\}$ called a *state*, modulo the following skein relations:



The algebra structure is given by stacking along the I-direction.

Theorem 29 (splitting map; [LÎ18, Thm. 3.1])**.** *Let Σ be a bordered, punctured surface, and let $c \subset \Sigma$ be an ideal arc. Let $\Sigma' = \Sigma \setminus c$ be the surface obtained by cutting Σ along c. Then, there is an algebra embedding*

$$\sigma : \mathrm{SkAlg}(\Sigma) \to \mathrm{SkAlg}(\Sigma')$$

given by

$$[L] \mapsto \sum_{\vec{\epsilon}} [L^{\vec{\epsilon}}],$$

where the sum is over $\{\pm\}^{L\cap c}$, i.e., all possible ways to assign states to $L \cap c$.

Proof. Any isotopy of L in $\Sigma \times I$ is a composition of finite sequence of elementary isotopies:

(1) An isotopy in the class of tangles in general position with respect to $c \times I$.
(2) Creation or annihilation of pairs of points in the intersection $L \cap (c \times I)$.
(3) Moving a crossing across $c \times I$ (i.e. height exchange).
(4) Moving a half-twist across $c \times I$.

One can check that under each of those elementary isotopies, the image doesn't change. Therefore, σ is well-defined.

For injectivity, we can isotope L so that the number of intersection points with $c \times I$ is minimal. Then, the "leading term" of $\sigma(L)$ (with respect to the grading given by sum of all states), which is $[L^{\vec{\epsilon}}]$ with $+$ on all the intersection points $L \cap c$, uniquely determines L. Then, for any non-zero $x = \sum_j c_j[L_j]$ with $c_j \neq 0$ and $[L_i] \neq [L_j]$ for any $i \neq j$, one can see that the leading term is non-zero, hence the image $\sigma(x)$ is also non-zero. ☐

Theorem 30 ([CL22a, Prop. 3.3])**.** *The stated skein algebra of a bigon D_2 is a Hopf algebra.*

Proof. The coalgebra structure is given by splitting a bigon into two bigons:

$$\Delta : \mathrm{SkAlg}(D_2) \to \mathrm{SkAlg}(D_2) \otimes \mathrm{SkAlg}(D_2).$$

It is easy to check that, together with the usual algebra structure, this makes $\mathrm{SkAlg}(D_2)$ a bialgebra. Comment on the existence of counit and antipode ☐

Definition 50. The Hopf algebra $\mathcal{O}_q(\mathrm{SL}_2)$, which is the Hopf dual of $U_q(\mathfrak{sl}_2)$, is generated by a, b, c, d with relations

$$ca = q\,ac, \quad db = q\,bd, \quad ba = q\,ab, \quad dc = q\,cd,$$
$$bc = cb, \quad ad - q^{-1}bc = 1, \quad da - q\,cb = 1,$$

with coproduct

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d, \quad \Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$

and counit

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0.$$

Its antipode is given by

$$S(a) = d, \quad S(d) = a, \quad S(b) = -q\,b, \quad S(c) = -q^{-1}c.$$

Theorem 31 ([CL22a, Thm 3.4])**.** *There is an isomorphism of Hopf algebras*

$$\phi : \mathrm{SkAlg}_A(D_2) \to \mathcal{O}_{q=A^2}(\mathrm{SL}_2)$$

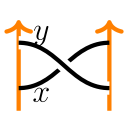
given on the generators by

$$\alpha_{+,+} \mapsto a$$
$$\alpha_{+,-} \mapsto b$$
$$\alpha_{-,+} \mapsto c$$
$$\alpha_{-,-} \mapsto d$$

Corollary 4. *For any choice of a boundary interval of a bordered punctured surface Σ, the stated skein algebra* $\mathrm{SkAlg}(\Sigma)$ *is a comodule over* $\mathcal{O}_{q=A^2}(\mathrm{SL}_2)$ *(or, equivalently, a module over* $U_{q=A^2}(\mathfrak{sl}_2)$*).*

Theorem 32 ([CL22a, Thm 3.5])**.** *The dual universal R-matrix $\rho \in (\mathcal{O}_{q=A^2}(\mathrm{SL}_2) \otimes \mathcal{O}_{q=A^2}(\mathrm{SL}_2))^*$ on $\mathcal{O}_{q=A^2}(\mathrm{SL}_2)$ can be described by*

$$\rho(x \otimes y) = \epsilon\left(\quad \right)$$



18. Lecture 18 (Tue Apr 9, 2024)

18.1. **Stated skein modules (cont.)**

18.1.1. *Stated skein algebras of surfaces (cont.)*

Definition 51. Let C be a coalgebra. For a C-C-bicomodule M, the (0-th) *Hochschild cohomology* $HH^0(M)$ is

$$HH^0(M) := \{m \in M \mid \Delta^l(m) = \mathrm{fl} \circ \Delta^r(m)\} \ \subset M,$$

where $\mathrm{fl} : x \otimes y \mapsto y \otimes x$ is the flip.

Theorem 33 ([CL22a, Thm. 4.8])**.** *In the setting of the splitting map (Theorem 29), the splitting map σ is an isomorphism onto the Hochschild cohomology of* $\mathrm{SkAlg}(\Sigma')$ *as a bicomodule over* $\mathcal{O}_{q=A^2}(\mathrm{SL}_2)$*:*

$$\sigma : \mathrm{SkAlg}(\Sigma) \xrightarrow{\sim} HH^0(\mathrm{SkAlg}(\Sigma')).$$

Remark 18. There is an analogous statement [CL22a, Thm. 4.10] where we view $\mathrm{SkAlg}(\Sigma')$ as a bimodule over $U_{q=A^2}(\mathfrak{sl}_2)$: the composition of the splitting map with the quotient to the 0-th Hochschild *homology* (which is a quotient of the bimodule) is an isomorphism.

18.1.2. *Reduced stated skein algebras.* [Reduced stated skein algebras, relation to 2d quantum trace map, following [CL22a] – To be written]

18.2. **Stated skein module of 3-manifolds.** [CL22b, PP24] [To be written]

18.3. **3d quantum trace map.** [PP24] [To be written]

19. Lecture 19 (Thu Apr 11, 2024)

19.1. **Non-semisimple quantum invariants.**

19.1.1. *Modified quantum dimension.* [Following [GPMT09, GKPM11] – To be written]

19.1.2. *Akutsu-Deguchi-Otsuki (ADO) invariants.* [To be written]

20. Lecture 20 (Tue Apr 16, 2024)

20.1. **Non-semisimple quantum invariants (cont.)**

20.1.1. *Relative G-modular categories.* [Following [CGPM14] – to be written]

20.1.2. *Costantino-Geer-Patureau (CGP) invariants.* [To be written]

21. Lecture 21 (Thu Apr 18, 2024)

22. Lecture 22 (Tue Apr 23, 2024)

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