# R-matrix and a q-series invariant of 3-manifolds

Sunghyuk Park

California Institute of Technology

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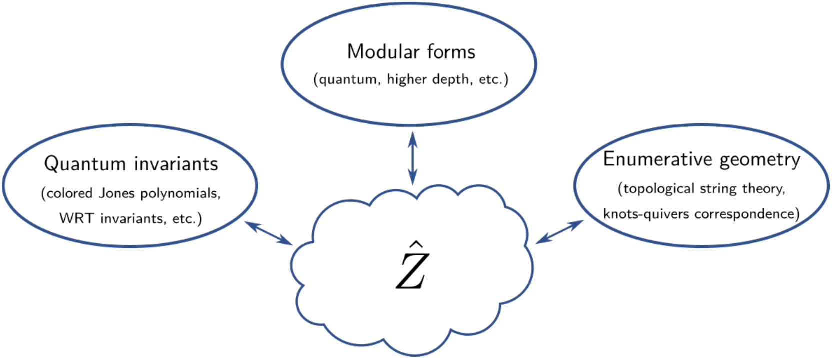
# Overview

A few years ago, S. Gukov and C. Manolescu conjectured that the Melvin-Morton-Rozansky expansion of the colored Jones polynomials can be resummed into a two-variable series $F_K(x, q)$, as part of a bigger program to construct a 3d TQFT $\hat{Z}$ that had been predicted from physics.

In this talk, I will explain how to prove their conjecture for a big class of links by "inverting" an R-matrix state sum.

This talk is based on [arXiv:2106.03942].

# Overview


Modular forms
(quantum, higher depth, etc.)
Quantum invariants
(colored Jones polynomials,
WRT invariants, etc.)
$\hat{Z}$
Enumerative geometry
(topological string theory,
knots-quivers correspondence)

# 1. Background and motivation

# Colored Jones polynomials

Let $L$ be a link and $n \geq 1$ an integer.

The $n$-colored Jones polynomial $J_{L,n}(q) \in \mathbb{Z}[q, q^{-1}]$ is an invariant of the link $L$ that can be defined in many different ways (e.g. Reshetikhin-Turaev construction, Jones-Wenzl projectors, cabling).

For our purposes, the definition using $U_q(\mathfrak{sl}_2)$ is the most useful.

# $U_q(\mathfrak{sl}_2)$

Quantum $\mathfrak{sl}_2$, $U_q(\mathfrak{sl}_2)$, is the associative algebra over $\mathbb{C}(q^{\frac{1}{2}})$ generated by $E, F, K(= q^{\frac{H}{2}}), K^{-1}$ with relations

$$KE = qEK, \quad KF = q^{-1}FK, \quad [E,F] = \frac{K - K^{-1}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

For each $n \geq 1$, let $V_n$ be the $n$-dimensional $U_q(\mathfrak{sl}_2)$-module with basis $\{v^0, \cdots, v^{n-1}\}$ on which the generators act by

$$Ev^j = [j]v^{j-1}, \quad Fv^j = [n-1-j]v^{j+1}, \quad Kv^j = q^{\frac{n-1-2j}{2}}v^j,$$

where $[m] := \frac{q^{\frac{m}{2}} - q^{-\frac{m}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$ for any $m \in \mathbb{Z}$.

# $U_q(\mathfrak{sl}_2)$

$U_q(\mathfrak{sl}_2)$ admits a universal $R$-matrix

$$R = q^{\frac{H\otimes H}{4}} \sum_{k\geq 0} q^{\frac{k(k-1)}{4}} \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^k}{[k]!} (E^k \otimes F^k) \in U_q(\mathfrak{sl}_2)\hat{\otimes} U_q(\mathfrak{sl}_2).$$

Applied to $V_n$, we obtain an automorphism $\check{R} \in \mathrm{Aut}(V_n \otimes V_n)$ given by

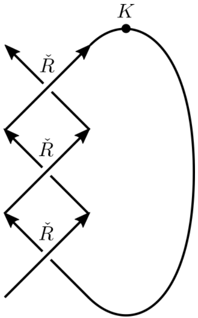
$$\check{R}(v^i \otimes v^j) = \sum_{k\geq 0} q^{\frac{n^2-1}{4} - \frac{(i-k+j+1)(n-1)}{2} + (i-k)j} \begin{bmatrix} i \\ k \end{bmatrix}_q \prod_{1\leq l\leq k} (1 - q^{-n+j+l}) v^{j+k} \otimes v^{i-k}$$

that satisfies the Yang-Baxter equation

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}.$$

# Colored Jones polynomials

Let $L$ be a link which can be presented as the closure of a braid $\beta$. Then the *n*-colored Jones polynomial $J_{L,n}(q)$ is the quantum trace of the automorphism given by the product of $R$-matrices.

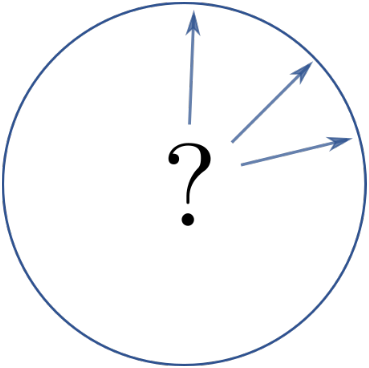


# Witten-Reshetikhin-Turaev (WRT) invariants

Let $Y$ be a closed 3-manifold and $\zeta_k$ a primitive $k$-th root of unity.

The WRT invariant $WRT_Y(\zeta_k)$ of $Y$ at $\zeta_k$ is a topological invariant of $Y$ that can be defined as a certain linear combination of $J_{L,n}(\zeta_k)$ for $1 \leq n \leq k-1$, if $Y$ is the result of the Dehn surgery on $L$.

# “Analytic continuation” of WRT invariants



# "Analytic continuation" of WRT invariants

**Theorem** ([Lawrence, Zagier (1999)])

*Let $P = \Sigma(2,3,5)$ be the Poincare homology sphere. Then*

$$WRT_P(\zeta_k) = \lim_{q \to \zeta_k} \frac{\hat{Z}_P(q)}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})},$$

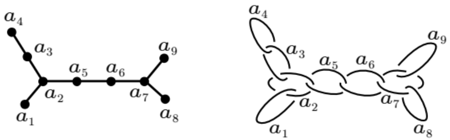
*where*

$$\hat{Z}_P(q) = q^{-\frac{3}{2}} \sum_{n \geq 0} \frac{(-1)^n q^{\frac{n(3n-1)}{2}}}{\prod_{1 \leq j \leq n}(1 - q^{n+j})}$$
$$= q^{-\frac{3}{2}}(1 - q - q^3 - q^7 + q^8 + q^{14} + \cdots).$$

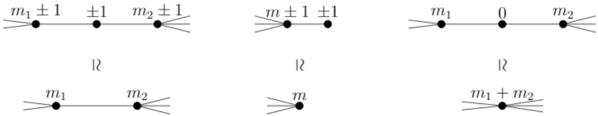
Similar results for more general Seifert fibered homology spheres: [Lawrence, Zagier (1999)], [Hikami (2005, 2006)]

# Plumbed 3-manifolds

For any tree Γ whose vertices are labelled by integers (called the plumbing graph), we can associate a 3-manifold, called the plumbed 3-manifold $Y_\Gamma$.




Neumann moves are Kirby moves for plumbing graphs.



# $\hat{Z}$

It turns out, for non-homology spheres, we get a number of $q$-series indexed by spin$^c$-structures.

**Definition ([Gukov, Pei, Putrov, Vafa (2017)])**

For a negative definite plumbed 3-manifold $Y_\Gamma$, define

$$\hat{Z}_{Y_\Gamma,b}(q) = \oint \prod_{v\in V} \frac{dx_v}{2\pi i x_v} \left( \prod_{v\in V} (x_v^{\frac{1}{2}} - x_v^{-\frac{1}{2}})^{2-\deg v} \sum_{\ell \in 2B\mathbb{Z}^V + b} q^{-\frac{1}{4}(\ell, B^{-1}\ell)} x^{\frac{\ell}{2}} \right),$$

where $B$ is the linking matrix of $\Gamma$, and

$$b \in (2\mathbb{Z}^V + \delta)/2B\mathbb{Z}^V \cong \mathrm{Spin}^c(Y_\Gamma).$$

This definition has various generalizations; see e.g. [Chung], [S.P.], [Ferrari, Putrov], [Akhmechet, Johnson, Krushkal].

# $\hat{Z}$

Some properties of $\hat{Z}$:

- It is invariant under Neumann moves and therefore is well-defined.
- It has integer coefficients.
- They provide examples of quantum modular forms (of possibly higher depth); see [Bringmann, Mahlburg, Milas], [Bringmann, Kaszian, Milas, Nazaroglu], [Zagier]

# $\hat{Z}$

The aforementioned definition of $\hat{Z}$ is motivated from physics, and it is expected that it can be extended to an invariant defined for *all* 3-manifolds.

An approach toward $\hat{Z}$ for general 3-manifolds:

1. study $\hat{Z}$ for link complements, and then
2. study the Dehn surgery formula.

This is the approach initiated in [Gukov, Manolescu].

# Melvin-Morton-Rozansky (MMR) expansion

**Theorem (Conjectured by [Melvin, Morton] and [Rozansky], proved by [Bar-Natan, Garoufalidis] and [Rozansky] (1990's))**

*Set $q = e^{\hbar}$. Consider the limit where $\hbar \to 0$ and $n \to \infty$ while $n\hbar$ is fixed. In this large-color limit, the colored Jones polynomial has the following expansion:*

$$J_{K,n}(q) = \frac{1}{\Delta_K(x)} + \frac{P_1(x)}{\Delta_K(x)^3}\hbar + \frac{P_2(x)}{\Delta_K(x)^5}\frac{\hbar^2}{2!} + \cdots ,$$

*where $x = q^n = e^{n\hbar}$, $\Delta_K(x)$ is the Alexander polynomial, and $P_j(x) \in \mathbb{Z}[x, x^{-1}]$ are Laurent polynomials invariant under Weyl symmetry $x \leftrightarrow x^{-1}$.*

# Gukov-Manolescu conjecture

**Conjecture ([Gukov, Manolescu (2019)])**

*The MMR expansion of the colored Jones polynomials can be resummed into a two-variable series*

$$(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{j \geq 0} \frac{P_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!} = F_K(x, q)$$

*where the left-hand side is understood as the average of its power series expansions near $x = 0$ and $x = \infty$, and $F_K(x, q)$ is a formal series of the form $\frac{1}{2} \sum_{m \in 2\mathbb{Z}+1} f_m(q) x^{\frac{m}{2}}$ with $f_m(q) \in \mathbb{Z}((q))$.*

*Moreover,*

$$\hat{A}_K(\hat{x}, \hat{y}, q) F_K(x, q) = 0,$$

*where $\hat{A}_K$ is the quantum A-polynomial for the unreduced colored Jones polynomials of $K$.*

# Dehn surgery formula

The two-variable series $F_K(x,q)$ is the $\hat{Z}$ for the knot complement $S^3 \setminus K$.

**Conjecture ([Gukov, Manolescu (2019)])**

$$\hat{Z}_{S^3_{p/r}(K),b}(q) = \oint \frac{dx}{2\pi i x}\left( (x^{\frac{1}{2r}} - x^{-\frac{1}{2r}}) F_K(x,q) \sum_{u \in \frac{p}{r}\mathbb{Z} + \frac{b}{r}} q^{-\frac{r}{p}u^2} x^u \right),$$

*provided that the right hand side converges.*

Note, this surgery formula is applicable only when $-\frac{r}{p}$ is big enough. A more general version of (conjectural) surgery formula is given in [S.P. (2021)].

# An example

$$F_{\mathbf{4}_1}(x,q) = -(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \sum_{n\geq 0} \frac{1}{\prod_{0\leq j\leq n}(x + x^{-1} - q^j - q^{-j})},$$

$$\hat{Z}_{-\Sigma(2,3,7)}(q) = \hat{Z}_{S^3_{-1}(\mathbf{4}_1)}(q) = \left[(x^{\frac{1}{2}} - x^{-\frac{1}{2}})F_{\mathbf{4}_1}(x,q)\right]_{x^u \mapsto q^{u^2}}$$
$$= \sum_{n\geq 0} \frac{q^{n^2}}{\prod_{1\leq k\leq n}(1 - q^{n+k})}.$$

This is Ramanujan's mock theta function $F_0(q)$ of order 7.

This is consistent with the fact that

$$\hat{Z}_{\Sigma(2,3,7)}(q) = \sum_{n\geq 0} \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{\prod_{1\leq k\leq n}(1 - q^{n+k})} = \text{“}\hat{Z}_{-\Sigma(2,3,7)}(q^{-1})\text{”}.$$

# 2. Large-color $R$-matrix and inverted state sums

# Large-color $R$-matrix

Since the MMR expansion is about the large-color asymptotics of the colored Jones polynomials, it is natural to study the corresponding large-color limit of the $R$-matrix.

In the large-color limit, the representation $V_n$ becomes a Verma module $V_\infty$ with generic highest (or lowest) weight.

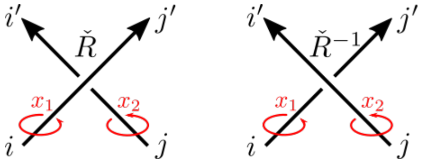
The highest weight Verma module with highest weight $\lambda = \log_q x - 1$ has a basis $\{v^j\}_{j\geq 0}$ on which the generators of $U_q(\mathfrak{sl}_2)$ act by

$$Ev^j = [j]v^{j-1}, \quad Fv^j = [\lambda - j]v^{j+1}, \quad Kv^j = q^{\frac{\lambda-2j}{2}}v^j.$$

# Large-color $R$-matrix

Call the $R$-matrix for these Verma modules the large-color $R$-matrix.

$$\check{R}(x_1, x_2)_{i,j}^{i',j'} = \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x_1^{\frac{-i'-j-1}{4}} x_2^{\frac{i-3j'-1}{4}} \begin{bmatrix} i \\ j' \end{bmatrix}_q \prod_{1\leq l\leq i-j'} (1 - q^{j+l} x_2^{-1})$$




Geometrically, $x_1$ and $x_2$ are the holonomy eigenvalues around the meridians of the two strands, in $SL_2(\mathbb{C})$ Chern-Simons theory at the abelian branch.

# Extending the $R$-matrix

While the indices $i, j, i', j' \in \mathbb{Z}_{\geq 0}$ represent the basis vectors $\{v^i\}_{i\geq 0}$ of the highest weight Verma module $V_\infty$, the $R$-matrix element $\check{R}(x_1, x_2)_{i,j}^{i',j'}$ makes sense for all $i, j, i', j' \in \mathbb{Z}$.

$$\check{R}(x,y)_{i,j}^{i',j'} =$$

$$\begin{cases} \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x^{-\frac{i'+j+1}{4}} y^{-\frac{3j'-i+1}{4}} \begin{bmatrix} i \\ i-j' \end{bmatrix}_q \prod_{1\leq l\leq i-j'} (1-q^{j+l} y^{-1}) & \text{if } \begin{matrix} i\geq j'\geq 0 \\ \text{or} \\ 0>i\geq j' \end{matrix} \\ \delta_{i+j,i'+j'} q^{(j+\frac{1}{2})(j'+\frac{1}{2})} x^{-\frac{i'+j+1}{4}} y^{-\frac{3j'-i+1}{4}} \begin{bmatrix} i \\ j' \end{bmatrix}_q \frac{1}{\prod_{0\leq l\leq j'-i-1}(1-q^{j-l}y^{-1})} & \text{if } j'\geq 0>i \\ 0 & \text{otherwise} \end{cases}$$

# Meaning of the extension

Consider the Verma module with height weight $\lambda = \log_q x - 1$

$$\cdots \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda - 2) \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda).$$

This can be naturally extended to the principal series module

$$\cdots \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda - 2) \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda) \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda + 2) \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda + 4) \underset{F}{\overset{E}{\rightleftharpoons}} \cdots .$$

The upper-half part of this module,

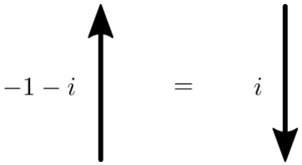
$$V_\infty(\lambda + 2) \underset{F}{\overset{E}{\rightleftharpoons}} V_\infty(\lambda + 4) \underset{F}{\overset{E}{\rightleftharpoons}} \cdots ,$$

is the lowest weight Verma module with lowest weight $\lambda + 2 = \log_q x + 1$.

# Meaning of the extension

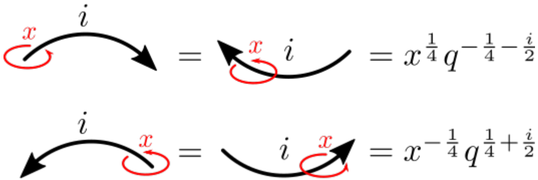
Therefore, changing the domain of an index from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{<0}$ corresponds to passing from the highest weight Verma module to the lowest weight Verma module.

There is a natural pairing between the highest and lowest weight Verma modules, so the "inversion" of the domain of an index can be thought of diagrammatically as changing the orientation of the segment of the link.



# Meaning of the extension

The pairing and copairing are given by

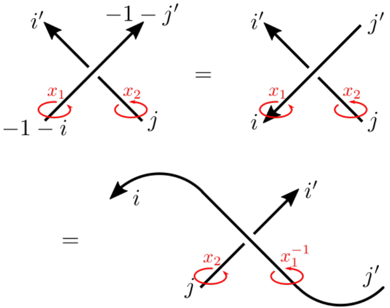


# Meaning of the extension

The large-color $R$-matrix enjoys various symmetries that are natural from the diagrammatic point of view. For instance,

$$\check{R}(x_1, x_2)^{i', -1-j'}_{-1-i, j} = q^{\frac{i-j'}{2}} \check{R}^{-1}(x_2, x_1^{-1})^{i, i'}_{j, j'},$$

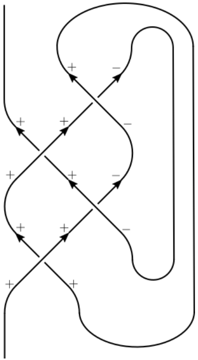
which can be diagrammatically understood as



# Inverted state sum

Let $L$ be an oriented link, and let $D$ be its diagram as a $(1,1)$-tangle.

An inversion datum on $D$ is a map that assigns a sign ($+$ or $-$) to each segment (arc) of $D$ in such a way that on each crossing the number of signs of each type are preserved.



# Inverted state sum

A diagram with an inversion datum determines a model of state sum that we call an inverted state sum.

That is, we replace each crossing by the corresponding $R$-matrix element and each local maximum and minimum in vertical direction by the corresponding pairing and copairing, respectively. Then we sum over all the internal indices; whenever the segment has $-$ sign, we flip range of summation to $\mathbb{Z}_{<0}$ (or equivalently, change the orientation of the segment).

Since this is an infinite sum, convergence is not always guaranteed.

# Inverted state sum

Let's say a link $L$ is "nice" if it admits a link diagram with an inversion datum such that the inverted state sum is absolutely convergent in $\mathbb{Z}[q, q^{-1}][[x^{-1}]]$.

**Theorem ([S.P. (2021)])**

*Gukov-Manolescu conjecture is true for any nice link.*

- Homogeneous braid links are nice.
- All fibered knots up to 10 crossings are nice.

**Corollary**

*For any 3-manifold $Y$, there is a link $L \subset Y$ for which we can compute $\hat{Z}_{Y \setminus L}(q)$. Therefore, the two-step approach toward $\hat{Z}$ for general 3-manifolds is reduced to the problem of finding a general Dehn surgery formula.*

# Connection to enumerative geometry

**Conjecture ([S.P. (2021)])**

*For any fibered knot $K$, the coefficients of $F_K(x,q)$ are in $\mathbb{Z}[q,q^{-1}]$.*

From enumerative geometry point of view, $F_K(x,a,q)$, a 1-parameter deformation of $F_K(x,q)$, is the count of open holomorphic curves in the resolved conifold with the knot complement Lagrangian. Knots-quivers correspondence for knot complements [Kucharski], [Ekholm, Gruen, Gukov, Kucharski, S.P., Stosic, Sulkowski] predicts that all such holomorphic curves arise from a finite collection of basic holomorphic disks.

When the knot is fibered, the knot complement Lagrangian can be shifted off from the zero section of $T^*S^3$, and by a positive area argument all the basic disk should have positive $x$-degree, implying that the coefficients of $F_K(x,a,q)$ as a power series in $x$ are rational functions in $a$ and $q$, just like the colored HOMFLYPT generating function.

The End

# Sketch of proof of the theorem

The main idea is to use Foata-Zeilberger formula.

A cycle on an oriented graph is called primitive if it is not a power of any other cycle. A multi-cycle is an unordered tuple of cycles. A multi-cycle is called primitive if each of the cycles are primitive.

**Theorem ([Foata, Zeilberger (1998)])**

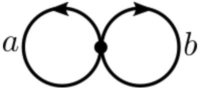
*Let $G$ be an oriented graph with weighted edges. Then the weighted sum of all primitive multi-cycles on $G$ is*

$$\frac{1}{\det(I-\mathcal{B})},$$

*where $\mathcal{B}$ denotes the transition matrix.*

# Sketch of proof of the theorem

Example:




$$
\begin{aligned}
&1\\
&+(a)+(b)\\
&+(ab)+(a)(b)+(a)^2+(b)^2\\
&+(a^2b)+(ab^2)+(a)(ab)+(b)(ab)+(a)^3+(a)^2(b)+(a)(b)^2+(b)^3\\
&+\cdots\\
&=\frac{1}{1-(a+b)}.
\end{aligned}
$$

# Sketch of proof of the theorem

In the classical limit $q = 1$, the inverted state sum becomes the weighted sum of all primitive multi-cycles on the oriented graph whose vertices are the set of internal segments. Therefore,

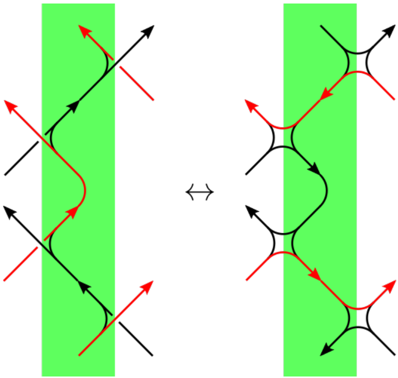
$$Z^{\mathrm{inv}}(D) = \frac{1}{\det(I - \mathcal{B}_{\mathrm{inv}})}.$$

The main lemma of the proof is to show that there is a nice one-to-one correspondence between the simple multi-cycles in the weighted oriented graphs before the inversion and after the inversion, that preserves the weight, up to a simple factor.

It follows that

$$\det(I - \mathcal{B}_{\mathrm{inv}}) = (-1)^s \det(I - \mathcal{B}) = (-1)^s \Delta_K(x).$$

# Sketch of proof of the theorem



# Sketch of proof of the theorem

To show that all the higher perturbative terms agree with the MMR expansion (up to the sign $(-1)^s$), we use the idea that Rozansky used in his proof of MMR conjecture.

That is, we use parametrized $R$-matrix. Higher perturbative terms of the $R$-matrix can be obtained by applying a differential operator to the parametrized $R$-matrix and then specializing the parameters.

The differential operator doesn't change upon extension of $R$-matrix, so it follows that the higher perturbative terms of the MMR expansion remains the same after inversion (up to the sign $(-1)^s$). This proves the theorem.