RESEARCH STATEMENT

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My research focuses on problems in geometry and topology, with particular emphasis on quantum topology and symplectic topology. I am especially interested in bringing ideas inspired by physics—such as quantum field theory and string theory—into a rigorous mathematical framework, using them to uncover new structures and deepen our understanding of low-dimensional topology.

Quantum topology is a vibrant field at the crossroads of topology, algebra, and representation theory, that studies low-dimensional spaces—such as three- and four-dimensional manifolds and knots—through the lens of rich algebraic structures like quantum groups, tensor categories, and higher categories. This field began with the discovery of the Jones polynomial, which revealed unexpected algebraic structures in knot theory and led to the development of quantum invariants of 3-manifolds, most notably through Witten's work on Chern-Simons theory and its mathematical incarnation in terms of quantum groups by Reshetikhin and Turaev. These ideas also inspired the formulation of topological quantum field theories (TQFTs), which encode topological data in algebraic terms, and culminated in the development of Khovanov homology—a categorification of the Jones polynomial that has led to striking advances in the study of smooth 4-manifolds, one of the most subtle and challenging frontiers in mathematics. A natural next step in this direction, and a major open problem in the field, is to extend the idea of categorification from knot invariants to 3-manifold invariants, providing a 3-manifold analog of Khovanov homology.

Symplectic topology is a central field in modern differential topology that studies spaces equipped with a symplectic structure—a geometric structure originally studied in classical mechanics to describe the evolution of physical systems. Though rooted in 19th-century mathematics, symplectic topology has undergone a transformative development in the late 20th century, most notably through Gromov's introduction of pseudo-holomorphic curves. This paved the way for powerful invariants such as Gromov–Witten invariants and Floer homology, which have since become indispensable tools in low-dimensional topology. While originally developed within symplectic topology, holomorphic curve techniques have recently begun to shed light on questions in quantum topology as well, revealing surprising connections between the two fields.

My work addresses foundational questions in quantum topology, while also exploring new connections with symplectic topology. One of the main focuses of my research has been the development of new quantum invariants of 3-manifolds, known as the "$\widehat{Z}$-invariants", as a step toward the categorification of quantum 3-manifold invariants. Another focus of my research has been "skein theory", the study of certain vector space-valued invariants of 3-manifolds, which exhibit deep and unexpected connections to hyperbolic geometry, symplectic topology, and Langlands duality. My work approaches skein theory both from the perspective of quantum topology and through holomorphic curve techniques. The remainder of this statement explores these aspects of my research in greater detail and outlines directions for future work.

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1. Quantum 3-manifold invariants and their categorification

A central open problem in quantum topology is to categorify quantum invariants of 3-manifolds—that is, to lift numerical quantum 3-manifold invariants to richer homological or categorical structures, much like how Khovanov homology [Kho00] lifts the Jones polynomial. Such a theory could offer deeper structural insight into 3-manifold topology and lead to powerful new invariants for smooth 4-manifolds, which remain among the most mysterious objects in mathematics.

Question 1. *Is there an analog of Khovanov homology for 3-manifolds?*

However, even after 25 years since the discovery of Khovanov homology, the problem of extending it to other 3-manifolds remains largely an open question, except in few cases, such as I-bundles over surfaces [APS04] and connected sums of copies of $S^2 \times S^1$ [Roz10, Wil21].

Nonetheless, insights from physics suggest that the answer to this question may still be "yes." In particular, recent developments in supersymmetric quantum field theory [Wit12, GPV17] predict the existence of categorified 3-manifold invariants, whose decategorified shadows are expected to be the so-called $\widehat{Z}$*-invariants*—a family of q-series-valued invariants introduced in [GPPV20]. These invariants thus offer a concrete, computable starting point for tackling the categorification problem.

Motivated by these predictions, my work brings the $\widehat{Z}$-invariants into the realm of quantum topology by providing a mathematical definition for them in a broad class of 3-manifolds. Previously, Gukov and Manolescu [GM21] defined the $\widehat{Z}$-invariants for negative definite plumbed 3-manifolds, including torus knot complements, and conjectured their extension to all knot complements. Building on this framework, I have proved their conjecture for a large class of knot complements by constructing the $\widehat{Z}$-invariants using state-sum techniques from the infinite-dimensional representation theory of quantum groups.

Theorem 1 ([Par20, Par21, Par22])**.** *To any "nice knot"—a class of knots that includes all closures of homogeneous braids—one can associate a two-variable power series, the* $\widehat{Z}$*-invariant of the knot complement, which defines a knot invariant. This invariant captures significant information: it encodes the Alexander polynomial as well as all colored Jones polynomials.*

The key ingredient in this construction is the *large-color R-matrix*, which describes the braiding of infinite-dimensional Verma modules of the quantum group. The natural next step toward the categorification problem is to categorify the large-color R-matrix itself. Recent work by Vaz, Naisse, and others on the categorification of Verma modules and their tensor products [LNV21, DN21] provides a promising foundation for this approach and may lead to a categorified version of the $\widehat{Z}$-invariants.

The structure of the $\widehat{Z}$-invariants exhibits striking parallels with Heegaard Floer homology: in both theories, the invariants decompose according to spin^c structures on the 3-manifold. This analogy is particularly clear for plumbed knot complements, where a combinatorial model for knot Floer homology—known as knot lattice homology—is available. In joint work with Akhmechet and Johnson, we proved the following:

Theorem 2 ([AJPar])**.** *There is an invariant of plumbed knot complements, called weighted bigraded roots, that unifies the* $\widehat{Z}$*-invariant and (the degree-zero part of) knot lattice homology. This invariant admits an explicit surgery formula.*

This parallel not only reveals structural connections between the two theories but also provides valuable clues toward the categorification of the $\widehat{Z}$-invariants.

Yet another promising approach toward categorifying the $\widehat{Z}$-invariants arises from their conjectural connection to Stokes phenomena in complex Chern–Simons theory. In recent work [GGMnW25], Garoufalidis and collaborators empirically observed that one can associate to a knot a matrix of q-series—interpreted as Stokes coefficients—with a distinguished entry given by the $\widehat{Z}$-invariant. These matrices are expected to admit a natural categorification via the Fukaya–Seidel category, or its combinatorial analog, the algebra of the infrared [GMW15, KKS16]. Related ideas have also appeared in the work of Gukov and Putrov [GP24], who proposed a way to categorify Stokes coefficients in Chern–Simons theory. This perspective offers a promising geometric framework for the categorification of the $\widehat{Z}$-invariants.

These developments have also created opportunities for undergraduate research within my program. In Spring 2022, I supervised a visiting undergraduate student on his senior thesis, focused on extending the large-color R-matrix to quantum $\mathfrak{so}_8$, building on my earlier work with quantum $\mathfrak{sl}_2$. More recently, in Summer 2025, I mentored two students through the Harvard Summer REU on a project investigating the infinite Jones–Wenzl projector, which provides a diagrammatic model for Verma modules. These projects introduced students to current ideas in quantum topology and representation theory, giving them meaningful exposure to ongoing research.

2. Skein theory and geometry

Skein theory provides a powerful framework for relating quantum invariants to geometric structures on surfaces and 3-manifolds. A key example of this connection is the realization that the SL_2-*skein algebra* of a surface Σ—an algebra generated by framed links in $\Sigma \times I$, modulo local relations reflecting the representation theory of quantum SL_2—provides a deformation quantization of the $\mathrm{SL}_2(\mathbb{C})$-character variety. At the same time, the *quantum Teichmüller space*—introduced by Chekhov–Fock [FC99] and Kashaev [Kas98]—quantizes the Teichmüller space, which corresponds to a connected component of the $\mathrm{PSL}_2(\mathbb{R})$-character variety. The work of Bonahon and Wong [BW11] clarified the relationship between these two quantizations by constructing the *quantum trace map*, a homomorphism from the skein algebra to the quantum Teichmüller space that quantizes the classical trace map between the corresponding character varieties. This map has had important applications, including the construction of irreducible representations of skein algebras at roots of unity [BW16, BW17], and has helped illuminate the link between quantum topology and geometric structures.

Building on these ideas, my research extends the foundational ideas underlying the quantum trace map into the 3-dimensional setting, developing new tools that connect skein modules, quantum invariants, and geometric structures on 3-manifolds. In recent joint work with Panitch, we constructed a 3-dimensional analog of the quantum trace map, relating the SL_2-skein module of a 3-manifold to its *quantum gluing module*—a quantization of Thurston's gluing variety, which parametrizes hyperbolic structures and forms a subspace of the $\mathrm{PSL}_2(\mathbb{C})$-character variety.

Theorem 3 ([PPar])**.** *There is a linear map—the 3d quantum trace map—from the* SL_2*-skein module of a 3-manifold to its quantum gluing module, that quantizes the classical trace map between the character varieties.*

Our work resolved a conjecture posed in [AGLR22] and led to a precise mathematical formulation of the *length conjecture*, which relates the colored Jones polynomials of a hyperbolic knot—decorated by a link in its complement—to both the link's hyperbolic length and a quantum invariant known as the *state integral.*

Beyond its conceptual significance, the 3d quantum trace map opens several promising directions for future research. One important application is to the study of skein modules specialized at roots of unity, where rich algebraic structures emerge. In the surface case, the 2d quantum trace map has been used to construct and classify representations of skein algebras in this setting by relating them to points in the character variety [BW16, BW17, FKBL19]. We hope to develop a similar theory in three dimensions: given a point in the $\mathrm{SL}_2(\mathbb{C})$-character variety of a 3-manifold, can one naturally associate a linear functional on its skein module at a root of unity? The 3d quantum trace map provides a new tool for approaching this question.

Another natural direction is to extend the construction to other gauge groups. In analogy with the SL_N-quantum trace map for surfaces [LY25], we are exploring a generalization of our 3d construction to SL_N-skein modules, with the goal of mapping to the SL_N-version of quantized gluing varieties studied by Dimofte, Gabella, and Goncharov [DGG16]. As in the SL_2 case, our approach involves decomposing the 3-manifold into simple pieces—such as ideal tetrahedra—and studying how skein modules behave under cutting and gluing, using additional boundary data to ensure compatibility across faces and edges.

An exciting direction of current work is the emerging relationship between the quantum trace map and a seemingly unrelated construction from physics: the *quantum UV–IR map* of Neitzke and Yan [NY20]. This map arises from the study of flat connections on Riemann surfaces and encodes how nonabelian structures—such as GL_2-local systems—can be abelianized on a branched cover known as the *spectral cover*. At a formal level, it defines a homomorphism from the GL_2-skein algebra of a surface to the GL_1-skein algebra of the cover. Though defined very differently from the quantum trace map, it was conjectured in [NY20]—for surfaces—that the two constructions are closely related. In ongoing joint work with Panitch, we prove this conjecture and also formulate and establish a natural generalization in three dimensions:

Theorem 4 (Panitch–P., in progress)**.** *The quantum trace map and the quantum UV–IR map are compatible in a way that is natural with respect to Pachner moves. In particular, for any knot complement, the quantum trace map can be recovered from the quantum UV–IR map.*

This compatibility suggests a promising path forward in higher rank. A conjectural extension of the quantum UV–IR map assigns to each element in the GL_N-skein algebra a GL_1-skein in a branched cover, but involves counting complicated flow graphs that are difficult to compute directly. In contrast, the SL_N-quantum trace map for 3-manifolds—once developed—may provide an algebraic and computable model for these maps.

3. Quantum topology from symplectic topology

Recent developments have revealed surprising and deep connections between quantum topology and symplectic topology. Physically, these connections are predicted by topological string theory: Witten [Wit95] proposed that Chern–Simons theory—the physical model underlying many quantum invariants—should emerge from string theory as a theory of holomorphic curves in a Calabi–Yau 3-fold. This idea has recently taken rigorous mathematical form through the work of Ekholm and Shende [ES25], who showed that open Gromov–Witten invariants, which count holomorphic curves in a Calabi–Yau 3-fold X with boundary on a Lagrangian 3-manifold $L \subset X$, can be organized as elements of the *HOMFLYPT skein module* $\mathrm{Sk}(L)$, a module generated by framed oriented links in L, modulo local relations reflecting those of HOMFLYPT polynomials. Remarkably, the boundary wall-crossing phenomena among holomorphic curves correspond precisely to these skein relations.

This perspective opens a two-way bridge between fields: skein-theoretic tools can shed light on holomorphic curve counts, and insights from symplectic geometry can, in turn, inform skein theory. In ongoing joint work with Ekholm, Longhi, and Shende, we are developing a framework for understanding how these curve counts behave under branched coverings of Lagrangians. Specifically, we consider a 3-manifold M and a Lagrangian submanifold $L \subset T^*M$ that defines a branched cover of M. For each link $K \subset M$, we study the skein-valued count of holomorphic curves with boundary on both L and the (shifted) conormal of K, producing an element $[K]_L \in \mathrm{Sk}(L)$. Surprisingly, the assignment $K \mapsto [K]_L$ depends only on the skein class of K in M. That is:

Theorem 5 (Ekholm–Longhi–P.–Shende, in progress)**.** *The map $K \mapsto [K]_L$ factors through the HOMFLYPT skein module of M, hence defining a linear map*

$$\mathrm{Tr} : \mathrm{Sk}(M) \to \mathrm{Sk}(L), \quad K \mapsto [K]_L.$$

This construction, which we call the *skein trace map*, exhibits striking and far-reaching properties. In particular, by specializing to double branched covers and passing to the GL_2-skein module of M and the GL_1-skein module of L, we recover the quantum UV–IR map of Neitzke and Yan [NY20], which was discussed in the previous section. Unlike the original construction, where isotopy invariance had to be verified by hand—a process that becomes increasingly intractable for branched covers of higher degree—our approach yields the well-definedness of the map from general principles of symplectic topology.

The skein trace map also reveals deep connections to the theory of Hecke algebras and symmetric functions. Remarkably, even in one of the simplest cases—the trivial double cover of the solid torus—this map encodes rich algebraic structure. In this setting, the (positive half of the) HOMFLYPT skein module of the solid torus is isomorphic to the Hopf algebra of symmetric functions [MS17], and the skein trace map agrees with its standard coproduct. More generally, for a twisted N-fold cover of the solid torus, we conjecture that the skein trace map corresponds to the transpose of the map induced on HOMFLYPT skein modules by the cabling of the solid torus by a torus link with N strands [MM08]. This line of inquiry opens up new avenues of research by uncovering previously unseen structure within HOMFLYPT skein modules and Hecke algebras.

Another promising direction, motivated by skein-valued curve counts and the skein trace map, is to develop a skein-theoretic analog of quantum cluster theory. In ordinary quantum cluster varieties—such as the quantum Teichmüller space discussed in the previous section—coordinate changes are governed by conjugation by the quantum dilogarithm, a function that satisfies both a pentagon identity and a 3-term recursion. In the holomorphic curve framework, the analog of this function is the *skein dilogarithm*—a skein-valued count of holomorphic disks—which satisfies similar algebraic identities [Nak24, Hu24]. One consequence of our construction of the skein trace map is that, under simple wall-crossings of spectral covers of surfaces, the associated skein trace maps before and after the wall-crossing are related by conjugation by the skein dilogarithm. This suggests the existence of a broader *skein-valued cluster theory*, in which mutation dynamics are controlled by holomorphic curve counts and skein recursion. Initial steps toward such a framework were taken by Hu, Schrader, and Zaslow [HSZ23]. Since holomorphic disks form only the simplest class of curves, such a theory would naturally encode a much richer algebraic structure, capturing the complexity of wall-crossing phenomena beyond the reach of classical cluster theory.

Finally, a particularly intriguing application of skein-valued curve counts connects them to the $\widehat{Z}$-invariants discussed in the first section. Given a fibered knot $K \subset S^3$, one can construct a Lagrangian submanifold in T^*S^3 diffeomorphic to the knot complement $S^3 \setminus K$, and count holomorphic curves with boundary on this Lagrangian and the zero section; fiberedness ensures that the Lagrangian can be shifted off the zero section to avoid intersection. Based on the dualities in string theory, these curve counts—after suitable specialization—are conjectured to reproduce the $\widehat{Z}$-invariant of $S^3 \setminus K$ [EGG+22]. Verifying this conjecture would give a geometric interpretation of the $\widehat{Z}$-invariants in terms of holomorphic curves, potentially opening a new path toward their categorification.

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