

\widehat{Z} AT LARGE N : FROM CURVE COUNTS TO QUANTUM MODULARITY

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Abstract

Reducing a 6d fivebrane theory on a 3-manifold Y gives a q -series 3-manifold invariant $\widehat{Z}(Y)$. We analyse the large- N behaviour of $F_K = \widehat{Z}(M_K)$, where M_K is the complement of a knot K in the 3-sphere, and explore the relationship between an a -deformed ($a = q^N$) version of F_K and HOMFLY-PT polynomials. On the one hand, in combination with counts of holomorphic annuli on knot complements, this gives an enumerative interpretation of F_K in terms of counts of open holomorphic curves. On the other, it leads to closed form expressions for a -deformed F_K for $(2, 2p + 1)$ torus knots and an order-by-order construction for other cases. They both suggest a further t -deformation based on superpolynomials, which can be used to obtain a t -deformation of ADO polynomials, expected to be related to categorification. Moreover, studying how F_K transforms under natural geometric operations on K indicates relations to quantum modularity in a new setting.

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1. INTRODUCTION

The motivation of this paper is an exploration of the vibrant intersection of low-dimensional topology, enumerative geometry, physics of BPS states, and three-dimensional cousins of elliptic genera with interesting modular properties. Focusing on finding new structures, we prove our statements for an infinite class of examples and analyse natural generalisations. Having in mind that we are making a first step out of many that are necessary to fully understand a new territory, we want to answer interrelated questions coming from the directions mentioned above. The first comes from the low-dimensional topology:

Question 1. *Is there an analogue of the HOMFLY-PT polynomial for 3-manifolds?*

Among knot invariants, the HOMFLY-PT polynomial plays a special role as it unifies A_{N-1} quantum group invariants of (super-)rank $N - 1$ for all values of N . For example, the Alexander polynomial, the Jones polynomial, and its higher-rank cousins are all specialisations of the HOMFLY-PT polynomial at $a = q^N$. Recent developments in 3d-3d correspondence suggest the existence of q -series invariants of 3-manifolds that play a role

similar to that of the Jones polynomial for knots. Then, a natural question is whether these new invariants exhibit regularity and stabilisation with respect to rank.

To explain our motivation from the perspective of enumerative invariants, it helps to recall different types of such invariants, summarised in Table 1. The revolution in enumerative geometry starts with the Gromov-Witten invariants that “count” stable maps $\phi : \Sigma_g \rightarrow X$ from a Riemann surface of genus g to the target manifold X , which for our discussion we assume to be a Calabi-Yau 3-fold. Topologically, such maps are classified by the genus g of Σ_g and by the homology class of its image, $\beta := \phi_*[\Sigma_g] \in H_2(X, \mathbb{Z})$. Then, the Gromov-Witten invariants of X are defined in terms of the intersection theory on the moduli space of stable maps, $\mathcal{M}_g(X, \beta)$, with fixed g and β . Although the resulting numerical invariants $\text{GW}_g(X, \beta)$ have the interpretation of “counting” stable maps, they are often rational rather than integer, because of denominators that account for automorphisms and resulting multi-valued perturbations.

| | Rational (\hbar) | Rational (q) | Integer | Refined |
|--------|--------------------------------------|---|--------------------------|-----------------------------|
| Closed | GW (stable maps) | Bare curves (constants not perturbed) | DT/GV (ideal sheaves) | Equivariant |
| Open | Open GW (relative stable maps) | Bare curves, boundary in skein | BPS invariants | Knot & 3-mfld homologies |

TABLE 1. Enumerative invariants.

We emphasize the non-integrality of Gromov-Witten invariants to help in comparing with many integer-valued enumerative invariants of X that play an important role in recent developments. The prominent examples are the Gopakumar-Vafa and Donaldson-Thomas invariants, which are close cousins and were independently discovered around the same time [DT98, GV98]. Much like Donaldson invariants of 4-manifolds [Don90], the Donaldson-Thomas invariants of X were originally formulated via analysis of six-dimensional gauge theory on X . In the modern literature, one often uses an equivalent formulation in terms of algebraic geometry of ideal sheaves \mathcal{I}_Z of subschemes $Z \subset X$, such that $\chi(\mathcal{O}_Z) = n$ and $[Z] = \beta \in H_2(X, \mathbb{Z})$. In particular, Donaldson-Thomas invariants of X are labelled basically by the same data as Gromov-Witten invariants (except that the genus g is replaced by the Euler characteristic n) and defined similarly, via integration over the virtual fundamental class of the moduli space. The resulting numerical invariants take integer values, $\text{DT}_n(X, \beta) \in \mathbb{Z}$, and, physically, have an interpretation as graded Euler characteristics of the spaces of BPS states, $\mathcal{H}_{*,n,\beta}^{\text{BPS}}(X)$. Conjecturally, these integer-valued invariants are connected with non-integer Gromov-Witten invariants via a universal relation that takes the same form for all X . Schematically, it looks like [MNOP06, GV98, Kat04]

$$(1) \quad \sum_n \text{DT}_n q^n \begin{array}{c} \xrightarrow{q = e^{\hbar} \rightarrow 1} \\ \xleftarrow{\text{Borel resum}} \end{array} \exp \left(\sum_m \text{GW}_m \hbar^m \right)$$

and involves two exponentials. One exponential appears on the right-hand side, replacing the formal series $\sum_m \text{GW}_m \hbar^m$ – which in general has zero radius of convergence and is often called Gromov-Witten potential – by its exponential. The second relates the variables on

the two sides, $q = e^{\hbar}$, and is one of the keys to integrality. As indicated in the third column of Table 1, there is a recently discovered [ES19] middle ground: counts of so-called bare curves, i.e. holomorphic curves with symplectic area zero components left unperturbed. Here the count is naturally in terms of $q = e^{\hbar}$ rather than \hbar itself, in a sense corresponding to the contribution to Gromov-Witten invariants of degree one in the Gopakumar-Vafa formula. Bare curves are also key to the understanding of open curve counts: boundaries of curves give line defects in Chern-Simons theory and open bare curves should be counted by the values of their boundary in the skein module of the brane where they end.

It is natural to ask whether the space $\mathcal{H}_{*,n,\beta}^{\text{BPS}}(X)$ itself contains more information than its Euler characteristic that yields $\text{DT}_n(X, \beta)$. This leads to the notion of *refined* BPS invariants, which in general can “jump” as one varies the complex structure of X . However, when X is rigid (e.g. toric), such refined invariants are well defined and indeed provide a more detailed information than Donaldson-Thomas (DT) or Gopakumar-Vafa (GV) invariants [KS10].

All enumerative invariants described so far can have an *open* analogue, which involves the data of the Calabi-Yau 3-fold X together with a Lagrangian submanifold $L \subset X$. The open Gromov-Witten invariants of (X, L) are then defined as count of (generalised) stable maps from *bordered* surfaces Σ , such that the boundary of Σ lands on L (for early work see e.g. [KL01, GZ02, LS01]). Just like closed GW invariants, their open cousins are \mathbb{Q} -valued and, based on physics predictions, should satisfy (1) with suitably defined integer *open* DT invariants. Unfortunately, the theory of open DT invariants and their refinements is not developed at present, except in a few special cases.

When X is toric and $L \cong S^1 \times \mathbb{R}^2$ is compatible with the torus action, one can compute (refined) open DT invariants of (X, L) via counting (skew) 3d partitions [AKMV05, ORV06, IKV09]. Another class of examples where an easily computable expression for open DT invariants was recently proposed involves (X, L) labelled by decorated graphs, the so-called plumbing graphs [GPPV17]. In yet another class of examples, related to knots, it was argued that refinement of open DT invariants is equivalent to the data of homological knot invariants [GSV05]. So, in principle, if one knows the latter, it can be used as a definition of refined open DT invariants at least in these special cases, until a better more universal definition is found.

Therefore, from the viewpoint of enumerative geometry, a challenge is to produce new families of examples where open DT invariants can be easily computed, either via combinatorial techniques, or via representation theory, that can hopefully shed light on the general case.

Question 2. *Can we find new easy-to-work-with definitions of open DT invariants, at least for special classes of (X, L) ?*

Our present work can be viewed as a step toward addressing this problem, for a particular choice of X and L , that also makes contact with recent developments of [DE20, ES19]. In particular, following [OV00, GSV05], we consider the enumerative geometry of HOMFLY-PT knot invariants, along with their refined/categorified analogues. We relate it to new q -series invariants of 3-manifolds mentioned earlier, thus presenting some evidence that Question 1 may have an affirmative answer. As pointed out in [GPPV17, sec.2.9], a similar relation between \widehat{Z} -invariants and enumerative geometry at finite N helps to understand the origin of Spin^c structures, which (non-canonically) can be identified with $b \in H_1(L, \mathbb{Z})$ and play a role similar to $\beta \in H_2(X, \mathbb{Z})$ in the closed case. Moreover, in the large- N limit we find a few more surprises that we summarise shortly in the form of new conjectures.

In finite rank N , the q -series invariants \widehat{Z} provide a non-perturbative definition of Chern-Simons theory with complex gauge group. They can also be viewed as $\mathcal{U}_q(\mathfrak{sl}_N)$ quantum group invariants for generic values of the parameter $|q| < 1$. Surprisingly, as a function of q , $\widehat{Z} = \sum_n \text{DT}_n q^n$ turns out to be either a character of a logarithmic vertex algebra, a Ramanujan mock modular theta function, or a q -series with integer coefficients and more exotic modular properties¹ that are expected to be a variant of quantum modularity [BMM20, CCF⁺19, BMM19, CFS19]. In order to study the large- N behaviour of these invariants, one needs powerful tools to compute them for any rank $N > 1$. Work in this direction was recently initiated in [Chu20, Par20a].

Studying the large- N behaviour of $\widehat{Z}(Y)$ for general 3-manifold Y is a very interesting but challenging problem. (See [GPV17, sec.7] for a brief survey of partial results in this direction and tools that could potentially be useful.) In this paper, we focus our attention on a simpler version of this problem by taking Y to be a knot complement $S^3 \setminus K$. Following [GM19, Par20a], we denote

$$(2) \quad F_K^{SU(N)}(x_1, \dots, x_{N-1}, q) := \widehat{Z}(S^3 \setminus K),$$

which depends on extra variables (x_1, \dots, x_{N-1}) that take values in the maximal torus of the complexified group $G_{\mathbb{C}} = SL(N, \mathbb{C})$. Even in this special case, the study of the large- N behaviour is highly nontrivial, and we hope to report on it in future work. For the purpose of the present paper, we specialise further to the case where only $x_1 \equiv qx \in \mathbb{C}^*$ is nontrivial and the rest are $x_i = q$ for $i \neq 1$. This corresponds to the choice of a one-dimensional subspace in the weight lattice of $G = SU(N)$ associated with symmetric representations. With this specialisation, which we denote by $F_K^{SU(N), \text{sym}}(x, q)$, we will be able to understand the large- N behaviour of (2). From the Chern-Simons theory perspective, the variable x is a holonomy eigenvalue along the meridian of the knot K and is one of the variables in the A -polynomial [Guk05]. It can also be understood as a variable in the Alexander polynomial $\Delta_K(x)$ [GM19]. Both of these polynomials will play an important role in our story. From the modularity viewpoint, x plays the role of a Jacobi-type variable.

The interpretation of \widehat{Z} as non-perturbative definition of Chern-Simons theory with complex gauge group (or 3d-3d correspondence) predicts that $F_K^{SU(N), \text{sym}}(x, q)$ should obey q -difference equations produced by the quantisation of character varieties. One of the main results in this paper, stated below in a form of more precise conjectures and theorems, is that both $F_K^{SU(N), \text{sym}}(x, q)$ and the q -difference equations themselves exhibit regularity with respect to N . Schematically,

$$(3) \quad \widehat{A}_K^{SU(N)}(\widehat{x}, \widehat{y}, q) F_K^{SU(N), \text{sym}}(x, q) = 0 \quad \Leftrightarrow \quad \widehat{A}_K(\widehat{x}, \widehat{y}, a, q) F_K(x, a, q) = 0,$$

where $a = q^N$, and $\widehat{A}_K(\widehat{x}, \widehat{y}, a, q)$ is the annihilator of coloured HOMFLY-PT invariants of K (the quantum a -deformed A -polynomial) [AV12, FGS13]. In more detail, we propose the following:

Conjecture 1 (a -deformed F_K). *For every knot $K \subset S^3$, there exists a three-variable function $F_K(x, a, q)$ interpolating all the $F_K^{SU(N), \text{sym}}$ in the following sense:*

¹A simple class of 3-manifolds for which the modular properties of $\widehat{Z}(M_3)$ have not yet been explicitly identified consists of surgeries on the figure-8 knot [GM19].

$$(4) \quad F_K(x, q^N, q) = F_K^{SU(N), sym}(x, q),$$

$$(5) \quad \hat{A}_K(\hat{x}, \hat{y}, a, q) F_K(x, a, q) = 0.$$

Moreover, it has the following properties:²

$$(6) \quad F_K(x, 1, q) = \Delta_K(x),$$

$$(7) \quad F_K(x, q, q) = 1,$$

$$(8) \quad \lim_{q \rightarrow 1} F_K(x, q^N, q) = \frac{1}{\Delta_K(x)^{N-1}}.$$

Its asymptotic expansion should agree with that of the coloured HOMFLY-PT polynomials. That is,

$$(9) \quad \log F_K(e^{\hbar}, a, e^{\hbar}) = \log P_r(K; a, e^{\hbar})$$

as \hbar -series.

For $(2, 2p + 1)$ torus knots it is possible to construct a -deformed F_K invariants basing on HOMFLY-PT polynomials (see Sections 5.2-5.3), which allows us to formulate the following:

Theorem 1. *Conjecture 1 is true for $(2, 2p + 1)$ torus knots. Specifically,*

$$(10) \quad F_{T(2, 2p+1)}(x, a, q) = \sum_{0 \leq k_p \leq \dots \leq k_1} x^{2(k_1 + \dots + k_p) - k_1} q^{(k_1 + k_2 + \dots + k_p) - \sum_{i=2}^p k_{i-1} k_i} \\ \times \frac{(aq^{-1}; q)_{k_1} (x; q^{-1})_{k_1}}{(q; q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \dots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}.$$

exhibits all properties of Conjecture 1.

For most knots we are not able to find an explicit formula for a -deformed F_K invariant, but for some we have a natural candidate:

Theorem 2. *If the abelian branch of the quantum a -deformed A -polynomial of a given knot is non-degenerate, the solution of the corresponding q -difference equation is unique and exhibits properties (5-8) of Conjecture 1.*

In consequence, as we show in Sections 5.4-5.5, we can solve the recursion encoded by the A -polynomial order by order and obtain a series expansion of the function that is expected to be an a -deformed F_K invariant.

In line with enumerative interpretations of the HOMFLY-PT polynomial in terms of counts of open holomorphic curves with boundary on the knot conormal in the resolved conifold, we give a similar enumerative interpretation of $F_K(a, q)$ in terms of counts of holomorphic curves with boundary on the knot complement. In the case of fibered knots, the knot complement Lagrangian in T^*S^3 can be shifted off of the zero section S^3 and then considered as a Lagrangian in the resolved conifold. Here the interpretation of $F_K(y, a, q)$ as a count of

²Here we are using the *reduced* normalisation. For the *unreduced* normalisation, we should have, for instance,

$$\lim_{q \rightarrow 1} F_K(x, q^N, q) = \left(\frac{x^{1/2} - x^{-1/2}}{\Delta_K(x)} \right)^{N-1},$$

curves with boundary in homology class $\log y$ is directly analogous to the knot conormal case (where the homology variable is $\log x$), and the classical limit ($q \rightarrow 1$ and $a \rightarrow 1$) was studied in [DE20]. For non-fibered knots the situation is further complicated by the appearance of intersection points between the knot complement Lagrangian and the zero section. Here, as in [ES19], we apply Symplectic Field Theory (SFT) stretching which leaves cotangent fibers in T^*S^3 . As we will discuss, these fibers are connected by Reeb chords at infinity that appear as negative ends in extra curves to be counted. For reasons of invariance of such counts, the values assigned to the Reeb chords are not arbitrary. They are functions of (x, a, q) determined in the semiclassical case by augmentations of a differential graded algebra and in the full quantum case by an analogous Legendrian SFT equation, in analogy with [EN18, Section 3]. Also, similar to [EN18, Section 6], the operator equation $\hat{A}_K = 0$ has an interpretation in terms of curve counts at the ideal boundary at infinity of the knot complement Lagrangian in T^*S^3 .

Since a -deformed quantum A -polynomials annihilating HOMFLY-PT polynomials turned out to be a great inspiration for the a -deformation of F_K invariants, it is natural to look at the t -deformation given by the quantum super- A -polynomials annihilating superpolynomials [FGS13]. We start from applying this idea to the Alexander polynomial $\Delta_K(x)$:

Definition 1. *A t -deformed Alexander polynomial is given by the $a = -t^{-1}$, $q = x$ specialisation of the superpolynomial:*

$$(11) \quad \Delta_K(x, t) = \mathcal{P}_K(-t^{-1}, x, t).$$

This allows for a consistent t -deformation of all properties of Conjecture 1:

Conjecture 2 ((a, t) -deformed F_K). *For every knot $K \subset S^3$, there exists a four-variable function $F_K(x, a, q, t)$ such that*

$$(12) \quad F_K(x, a, q, -1) = F_K(x, a, q),$$

$$(13) \quad \hat{A}_K(\hat{x}, \hat{y}, a, q, t) F_K(x, a, q, t) = 0.$$

Moreover, it has the following properties:

$$(14) \quad F_K(x, -t^{-1}, q, t) = \Delta_K(x, t),$$

$$(15) \quad F_K(x, -t^{-1}q, q, t) = 1,$$

$$(16) \quad \lim_{q \rightarrow 1} F_K(x, -t^{-1}q^N, q, t) = \frac{1}{\Delta_K(x, t)^{N-1}}.$$

Its asymptotic expansion should agree with that of the superpolynomials. That is,

$$(17) \quad \log F_K(e^{r\hbar}, a, e^{\hbar}, t) = \log \mathcal{P}_r(K; a, e^{\hbar}, t)$$

as \hbar -series.

The construction of the a -deformed F_K invariants for $(2, 2p + 1)$ torus knots allows for a natural generalisation based on the superpolynomials (see Section 6.1), which leads to the following:

Theorem 3. *Conjecture 2 is true for $(2, 2p + 1)$ torus knots. Specifically,*

$$(18) \quad F_{T(2,2p+1)}(x, a, q, t) = \sum_{0 \leq k_p \leq \dots \leq k_1} x^{2(k_1 + \dots + k_p) - k_1} q^{(k_1 + k_2 + \dots + k_p) - \sum_{i=2}^p k_{i-1} k_i} t^{2(k_1 + \dots + k_p)} \\ \times \frac{(-aq^{-1}t; q)_{k_1} (x; q^{-1})_{k_1}}{(q; q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}.$$

exhibits all properties of Conjecture 2.

Having discussed the a - and t -deformation, let us pause and think about the variable x . For the trefoil knot, the most basic $SU(2)$ version of F_K is itself a “deformation” of the Dedekind eta-function:³

$$(19) \quad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}} \quad \rightsquigarrow \quad F_{3_1} = \sum_{m=1}^{\infty} \epsilon_m q^{\frac{m^2}{24}} (x^m - x^{-m}).$$

Inspired by [Zag10], we may ask:

Question 3. *What are the modular properties of $F_K(x, a, q)$? Is there a number theory (or vertex algebra) interpretation of the q -difference equations (3)?*

Hoping that future work will shed light on the first part of the question, in this paper we will only offer some clues regarding the second part. For example, one important lesson that follows from the general framework reviewed in the introduction and used in the rest of this paper is that the q -difference equation for the mirror knot $m(K)$ is related to that of K simply by replacing $q \mapsto q^{-1}$, $a \mapsto a^{-1}$ (and $t \mapsto t^{-1}$ in the refined case). Since $\hat{A}_K(\hat{x}, \hat{y}, a, q)$ is a rational function of these variables, it transforms in a simple way under $K \mapsto m(K)$. On the other hand, the solutions to the corresponding q -difference equation, $F_K(x, a, q)$ and $F_{m(K)}(x, a, q)$, are related in a highly nontrivial way, so we expect generalisations of variants of quantum modularity found in [BMM20, CCF⁺19, BMM19, CFS19]. This is interesting also for the interpretation of $F_K(x, a, q)$ and $F_{m(K)}(x, a, q)$ as characters of “dual” logarithmic vertex algebras, where it gives a nice structural property shared by completely different vertex operator algebras.

The rest of the paper is organised as follows. In Section 2 we introduce F_K invariants (\widehat{Z} invariants for knot complements). Their relations with low-dimensional topology, physics, and enumerative geometry are presented in Sections 3 and 4. Section 5 contains proofs of Theorems 1 and 2 together with examples, whereas Section 6 describes a t -deformation, including a proof of Theorem 3. In Section 7 we show how the analysis of the behaviour of F_K invariants under taking the mirror of the knot $K \rightarrow m(K)$ provides a new area for studies of the quantum modularity. Finally, in Section 8 we discuss interesting problems for future research.

2. \widehat{Z} AND F_K INVARIANTS

In their study of 3d $\mathcal{N} = 2$ theory $T[Y]$ for 3-manifolds Y , Gukov-Putrov-Vafa [GPV17] and Gukov-Pei-Putrov-Vafa [GPPV17] conjectured the existence of the 3-manifold invariants $\widehat{Z}(Y)$

³In this “deformation” the m -th term is multiplied by $x^m - x^{-m}$. Another version, that appears e.g. in [Par20b], is when the m -th term is multiplied just by x^m ; it gives a genuine deformation of the η -function, in which the latter is recovered by taking $x \rightarrow 1$ and is related to the invariant $F_K(x, q)$ by further anti-symmetrisation with respect to $x \rightarrow x^{-1}$.

(also known as “homological blocks” or “GPPV invariants”) valued in q -series with integer coefficients. These q -series invariants exhibit peculiar modular properties, the exploration of which was initiated in [BMM20, CCF⁺19, BMM19, CFS19].

More recently, Gukov-Manolescu [GM19] introduced a version of \widehat{Z} for knot complements, which they called F_K : if $K \subset S^3$ is a knot, then $F_K = \widehat{Z}(S^3 \setminus K)$. The motivation was to study \widehat{Z} more systematically using Dehn surgery. Recall that the Melvin-Morton-Rozansky expansion [MM95, BNG96, Roz96, Roz98] (also known as “loop expansion” or “large colour expansion”) of the coloured Jones polynomials is the asymptotic expansion near $\hbar \rightarrow 0$ while keeping $x = q^r = e^{r\hbar}$ fixed:

$$(20) \quad J_r(K; q = e^{\hbar}) = \sum_{j \geq 0} \frac{p_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}, \quad p_j(x) \in \mathbb{Z}[x, x^{-1}], \quad p_0 = 1.$$

The main conjecture of [GM19] was then the following:

Conjecture 3. *For every knot $K \subset S^3$, there exists a two-variable series*

$$(21) \quad F_K(x, q) = \frac{1}{2} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} f_m(q)(x^{m/2} - x^{-m/2}), \quad f_m(q) \in \mathbb{Z}[q^{-1}, q]$$

such that its asymptotic expansion agrees with the Melvin-Morton-Rozansky expansion of the coloured Jones polynomials⁴:

$$(22) \quad F_K(x, q = e^{\hbar}) = (x^{1/2} - x^{-1/2}) \sum_{j \geq 0} \frac{p_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}.$$

Moreover, this series is annihilated by the quantum A -polynomial:

$$(23) \quad \hat{A}_K(\hat{x}, \hat{y}, q) F_K(x, q) = 0.$$

Conjecture 3 concerns $G = SU(2)$. An extension to arbitrary G was studied in [Par20a]. In particular, the existence of an $SU(N)$ generalisation of F_K , denoted $F_K^{SU(N)}$, was conjectured and it was observed that its specialisation to symmetric representations, $F_K^{SU(N), sym}$, is annihilated by the corresponding quantum A -polynomial:

$$(24) \quad \hat{A}_K(\hat{x}, \hat{y}, a = q^N, q) F_K^{SU(N), sym}(x, q) = 0.$$

From this perspective it is natural to ask about the large- N behaviour and existence of an a -deformed (i.e. HOMFLY-PT analogue of) the F_K invariant. This was the starting point of this paper and, as we will see, HOMFLY-PT analogues of F_K indeed exist.

3. RELATIONS WITH LOW-DIMENSIONAL TOPOLOGY AND PHYSICS

In this section we discuss the connection between F_K and other knot invariants, such as HOMFLY-PT and A -polynomials, and 3d $\mathcal{N} = 2$ effective theories engineered on the world-sheets of M5-branes.

⁴[GM19] uses the unreduced normalisation. In the reduced normalisation, used in the major part of this paper, (22) reads $F_K(x, q = e^{\hbar}) = \sum_{j \geq 0} \frac{p_j(x)}{\Delta_K(x)^{2j+1}} \frac{\hbar^j}{j!}$.

3.1. HOMFLY-PT polynomials. If $K \subset S^3$ is a knot, then its HOMFLY-PT polynomial $P(K; a, q)$ is a topological invariant [HOM⁺85, PT87] which can be calculated via the skein relation:

$$(25) \quad a^{1/2}P(\times) - a^{-1/2}P(\oslash) = (q^{1/2} - q^{-1/2})P(\cup)$$

with normalisation condition $P(0_1; a, q) = 1$. This is called the *reduced* normalisation and corresponds to dividing by the full natural HOMFLY-PT polynomial for the unknot (here denoted by bar):

$$(26) \quad \begin{aligned} P(K; a, q) &= \frac{\bar{P}(K; a, q)}{\bar{P}(0_1; a, q)}, \\ \bar{P}(0_1; a, q) &= \frac{a^{1/2} - a^{-1/2}}{q^{1/2} - q^{-1/2}}. \end{aligned}$$

We use the reduced normalisation in the major part of the paper, except the geometric considerations in Sections 4 and 7, where we analyse curve counts leading to fully unreduced normalisation – corresponding to $\bar{P}(K; a, q)$ – and explain how to obtain the reduced one.

More generally, the coloured HOMFLY-PT polynomials $P_R(K; a, q)$ are similar polynomial knot invariants depending also on a representation R of $SU(N)$. In this setting, the original HOMFLY-PT corresponds to the defining representation. We will be interested mainly in HOMFLY-PT polynomials coloured by the totally symmetric representations S^r with r boxes in one row of the Young diagram. In order to simplify the notation, we will denote them by $P_r(K; a, q)$ and call HOMFLY-PT polynomials.

From our perspective, the most important property of HOMFLY-PT polynomials is the fact that, after the substitution $a = q^N$, they reduce to the $SU(N)$ coloured Jones polynomials:

$$(27) \quad P_r(K; q^N, q) = J_r^{SU(N)}(K; q),$$

whose asymptotic expansion (as a series in \hbar) agrees with symmetric $SU(N)$ F_K invariants [Par20a, GM19]. This connection can be considered as the base of the relation between HOMFLY-PT polynomials and a -deformed F_K invariants.

In [DGR06, GS12a], a t -deformation of HOMFLY-PT polynomial was proposed. The super-polynomial $\mathcal{P}_r(K, a, q, t)$ was defined as a Poincaré polynomial of the triply-graded homology that categorifies the HOMFLY-PT polynomial:

$$(28) \quad \begin{aligned} P_r(K; a, q) &= \sum_{i,j,k} (-1)^k a^i q^j \dim \mathcal{H}_{i,j,k}^{S^r}(K), \\ \mathcal{P}_r(K; a, q, t) &= \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}_{i,j,k}^{S^r}(K). \end{aligned}$$

We will use this categorification to propose an (a, t) -deformed F_K invariant in Section 6.

3.2. A -polynomials. The A -polynomial is a knot invariant associated to a character variety of a complement of a given knot K in S^3 [CCG⁺94]. It takes the form of an algebraic curve $A_K(x, y) = 0$, for $x, y \in \mathbb{C}^*$. According to the volume conjecture, it also captures the asymptotics of coloured Jones polynomials $J_r(K; q)$ for large colours r . The quantisation of the A -polynomial encodes information about all colours, not only large. Namely, it gives the recursion relations satisfied by $J_r(K; q)$, which can be written in the form

$$(29) \quad \hat{A}_K(\hat{x}, \hat{y}) J_*(K; q) = 0,$$

where \hat{x}, \hat{y} act by

$$(30) \quad \hat{x}J_r = q^r J_r, \quad \hat{y}J_r = J_{r+1},$$

and satisfy the relation $\hat{y}\hat{x} = q\hat{x}\hat{y}$. The above conjecture was proposed independently in the context of quantisation of Chern-Simons theory [Guk05] and in parallel mathematics developments [Gar04]. The operator $\hat{A}_K(\hat{x}, \hat{y})$ is referred to as the quantum A -polynomial; in the limit $q = 1$ it reduces to the polynomial defining the A -polynomial algebraic curve $A_K(x, y)$.

The above conjectures were generalised to coloured HOMFLY-PT polynomials [AV12] and coloured superpolynomials [AGSF12, FGS13], which we introduced in (28). In these cases the objects mentioned in the previous paragraph become a - and t -dependent. In particular, the asymptotics of coloured superpolynomials $\mathcal{P}_r(K; a, q, t)$ for large r is captured by an algebraic curve called super- A -polynomial, defined by an equation $A_K(x, y, a, t) = 0$. For $t = -1$ it reduces to a -deformed A -polynomial, and upon setting in addition $a = 1$ we obtain the original A -polynomial as a factor. For brevity, all these objects are often referred to as A -polynomials. A quantisation of the super- A -polynomial gives rise to quantum super- A -polynomial $\hat{A}_K(\hat{x}, \hat{y}, a, q, t)$, which is an operator that imposes recursion relations for coloured superpolynomials:

$$(31) \quad \hat{A}_K(\hat{x}, \hat{y}, a, q, t)\mathcal{P}_*(K; a, q, t) = 0.$$

A universal framework that enables to determine a quantum A -polynomial from an underlying classical curve $A(x, y) = 0$ was proposed in [GS12b] (irrespective of extra parameters these curves depend on, and also beyond examples related to knots).

The quantum A -polynomial $\hat{A}_K(\hat{x}, \hat{y})$ can be regarded as a polynomial in \hat{y} , whose coefficients depend on q and $x = q^r$. It was conjectured in [GM19] that at the same time, the quantum A -polynomial is an operator that annihilates F_K , once \hat{x} is interpreted as a multiplication by x and \hat{y} acts by $\hat{y}F_K(x, q) = F_K(qx, q)$. While the same form of quantum A -polynomial $\hat{A}_K(\hat{x}, \hat{y})$ arises in the analysis of coloured Jones polynomial and F_K invariants, there is a subtle but important difference between these two situations, which has to do with the initial conditions that need to be imposed.

One of the main results of this paper is the statement that (a, t) -deformed F_K invariants are annihilated by quantum super- A -polynomial $\hat{A}_K(\hat{x}, \hat{y}, a, q, t)$ that we presented above (or its $t = -1$ limit in case of a -deformed F_K invariants). We verify this statement for the family of $(2, 2p + 1)$ torus knots and figure-eight knot. Apart from the conceptual importance, this statement implies that we can simply take advantage of expressions for quantum super- A -polynomials derived before, e.g. in [FGS13, FGSS13]. Nonetheless, to determine F_K using these quantum A -polynomials – or to check that they are indeed annihilated by $\hat{A}_K(\hat{x}, \hat{y}, a, q, t)$ – we need to use proper initial conditions. Our conventions in this paper are such that, in comparison with [FGS13], we rescale \hat{x} by q . Additionally we rescale \hat{y} by a/q which has the effect of removing a common prefactor of $x^{\frac{\log(a)}{h}-1}$ from F_K . We discuss all these issues in detail in the following sections.

3.3. 3d-3d correspondence. From the physical point of view, the \widehat{Z} -invariants of a 3-manifold Y encode the BPS spectrum of N fivebranes supported on $\mathbb{R}^2 \times S^1 \times Y$, where Y is

embedded (as a zero-section) inside the Calabi-Yau space T^*Y and $\mathbb{R}^2 \times S^1 \subset \mathbb{R}^4 \times S^1$:

$$(32) \quad \begin{array}{l} \text{space-time : } \mathbb{R}^4 \times S^1 \times T^*Y \\ \cup \qquad \qquad \cup \\ N \text{ M5-branes : } \mathbb{R}^2 \times S^1 \times Y. \end{array}$$

Taking the large- N limit of this system for general Y is highly nontrivial (see [GPV17, sec.7] and [ES19, Remark 2.4]). However, when Y is a knot complement $M_K := S^3 \setminus K$ then there is an equivalent description of the physical system (32) for which the study of large- N behaviour can be reduced to the celebrated ‘‘large- N transition’’ [GV98, OV00].

Note that from the viewpoint of 3d-3d correspondence, N fivebranes on $Y = M_K$ produce a 4d $\mathcal{N} = 4$ theory – which is a close cousin of 4d $\mathcal{N} = 4$ SYM but is *not* 4d $\mathcal{N} = 4$ SYM – on a half-space $\mathbb{R}^3 \times \mathbb{R}_+$ coupled to 3d $\mathcal{N} = 2$ theory $T[M_K]$ on the boundary. Indeed, near the boundary $T^2 = \Lambda_K = \partial M_K$, the compactification of N fivebranes produces a 4d $\mathcal{N} = 4$ theory which has moduli space of vacua $\text{Sym}^N(\mathbb{C}^2 \times \mathbb{C}^*)$ [CGPS19]. (Recall that the moduli space of vacua in 4d $\mathcal{N} = 4$ super-Yang-Mills is $\text{Sym}^N(\mathbb{C}^3)$.) The $SU(N)$ gauge symmetry of this theory appears as a global symmetry of the 3d boundary theory $T[M_K]$. In particular, the variables $x_i \in \mathbb{C}^*$ in (2) are complexified fugacities for this global (‘‘flavour’’) symmetry. For $G = SU(2)$, the moduli space of vacua of the knot complement theory $T[M_K]$ gives precisely the A -polynomial of K . And, similarly, $G_{\mathbb{C}}$ character varieties of M_K are realised as spaces of vacua in $T[M_K, SU(N)]$ with $G = SU(N)$ [FGS13, FGSS13].

Now, as promised, let us give another, equivalent description of the physical system (32) with $Y = M_K$, where the large- N behaviour is easier to analyse:

$$(33) \quad \begin{array}{l} \text{space-time : } \mathbb{R}^4 \times S^1 \times T^*S^3 \\ \cup \qquad \qquad \cup \\ N \text{ M5-branes : } \mathbb{R}^2 \times S^1 \times S^3 \\ \rho \text{ M5-branes : } \mathbb{R}^2 \times S^1 \times L_K. \end{array}$$

This brane configuration is basically a variant of (32) with $Y = S^3$ and ρ extra M5-branes supported on $\mathbb{R}^2 \times S^1 \times L_K$, where $L_K \subset T^*S^3$ is the conormal bundle of the knot $K \subset S^3$. There is, however, a crucial difference between fivebranes on S^3 and L_K . Since the latter are non-compact in two directions orthogonal to K , they carry no dynamical degrees of freedom away from K . One can path integrate those degrees of freedom along K ; this effectively removes K from S^3 and puts the corresponding boundary conditions on the boundary $T^2 = \partial M_K$. The resulting system is precisely (32) with $Y = M_K$. Equivalently, one can use the topological invariance along S^3 to move the tubular neighbourhood of $K \subset S^3$ to ‘‘infinity.’’ This creates a long neck $\cong \mathbb{R} \times T^2$ as in the above discussion. Either way, we end up with a system of N fivebranes on the knot complement and no extra branes on L_K , so that the choice of $GL(\rho, \mathbb{C})$ flat connection on L_K is now encoded in the boundary condition for $SL(N, \mathbb{C})$ connection⁵ on $T^2 = \partial M_K$. In particular, the latter has at most ρ nontrivial parameters $x_i \in \mathbb{C}^*$, $i = 1, \dots, \rho$.

Although the relation between N fivebranes on a knot complement and (33) holds for any value of ρ (with a suitable identification of boundary conditions, of course), the extreme values are somewhat special. The maximal value $\rho = N$ is what one needs to study the full-fledged

⁵To be more precise, it is a $GL(N, \mathbb{C})$ connection, but the dynamics of the $GL(1, \mathbb{C})$ sector is different from that of the $SL(N, \mathbb{C})$ sector and can be decoupled.

\widehat{Z} -invariants, cf. (2). However, this, or any other choice of $\rho \sim O(N)$ make the study of $N \rightarrow \infty$ rather challenging since both sets of fivebranes in (33) need to be replaced by geometry and such generalised “geometric transitions” are not known. The other extreme is when $\rho \sim O(1)$ as $N \rightarrow \infty$; in particular, in this paper we consider the simplest such option $\rho = 1$. In that case we can use the geometric transition of Gopakumar and Vafa [GV98], upon which there is one brane on L_K and N fivebranes on the zero-section of T^*S^3 disappear. Then the Calabi-Yau space T^*S^3 undergoes a topology changing transition to X , the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{C}\mathbf{P}^1$, the so-called “resolved conifold.” Only the Ooguri-Vafa fivebranes supported on the conormal bundle L_K remain:

$$(34) \quad \begin{array}{l} \text{space-time : } \mathbb{R}^4 \times S^1 \times X \\ \qquad \qquad \qquad \cup \qquad \qquad \cup \\ \rho \text{ M5-branes : } \mathbb{R}^2 \times S^1 \times L_K. \end{array}$$

Note that on the resolved conifold side, i.e. after the geometric transition, $\log a = \text{Vol}(\mathbb{C}\mathbf{P}^1) + i \int B = N\hbar$ is the complexified Kähler parameter which, as usual, enters the generating function of enumerative invariants presented in Table 1.

To summarise, a system of N fivebranes on a knot complement (32) is equivalent to a brane system (34), with a suitable map that relates the boundary conditions in the two cases. There is another system, closely related to (34), that one can obtain from (33) by first reconnecting ρ branes on L_K with ρ branes on S^3 . This gives ρ branes on M_K (that go off to infinity just like L_K does) plus $N - \rho$ branes on S^3 . Assuming that $\rho \sim O(1)$ as $N \rightarrow \infty$ (e.g. $\rho = 1$ in the context of this paper), after the geometric transition we end up with a system like (34), except L_K is replaced by M_K and $\text{Vol}(\mathbb{C}\mathbf{P}^1) + i \int B = (N - \rho)\hbar$. Both of these systems on the resolved side compute the HOMFLY-PT polynomials of K coloured by Young diagrams with at most ρ rows.

The leading, genus-0 contribution to the generating function of enumerative invariants is the twisted superpotential. It can be computed either on the resolved side of the transition, where a is a Kähler parameter, or on the original (“deformed”) side, for a family of theories labelled by N . Either way, one finds that the twisted superpotential is given by the double-scaling limit that combines large-colour and semiclassical limits of the HOMFLY-PT polynomials [FGS13, FGSS13]:

$$(35) \quad P_r(K; a, q) \xrightarrow[\hbar \rightarrow 0]{r \rightarrow \infty} \int \prod_i \frac{dz_i}{z_i} \exp \left[\frac{1}{\hbar} \widetilde{\mathcal{W}}(z_i, x, a) + O(\hbar^0) \right],$$

with $x = q^r$ kept fixed. We can read off the structure of $T[M_K]$ from the terms in $\widetilde{\mathcal{W}}(z_i, x, a)$:

$$(36) \quad \begin{array}{ll} \text{Li}_2(a^{n_Q} x^{n_M} z_i^{n_i}) & \longleftrightarrow \quad (\text{chiral field}), \\ \frac{\kappa_{ij}}{2} \log \zeta_i \cdot \log \zeta_j & \longleftrightarrow \quad (\text{Chern-Simons coupling}). \end{array}$$

Each dilogarithm is interpreted as the one-loop contribution of a chiral superfield with charges (n_Q, n_M, n_i) under the global symmetries $U(1)_Q$ (arising from the internal 2-cycle in X) and $U(1)_M$ (corresponding to the non-dynamical gauge field on the M5-brane), and the gauge group $U(1) \times \dots \times U(1)$. Quadratic-logarithmic terms are identified with Chern-Simons couplings among the various $U(1)$ symmetries, with ζ_i denoting the respective fugacities.

We can integrate out the dynamical fields (whose VEVs are given by $\log z_i$) using the saddle point approximation to obtain the effective twisted superpotential:

$$(37) \quad \widetilde{\mathcal{W}}_{\text{eff}}(x, a) = \frac{\partial \widetilde{\mathcal{W}}(z_i, x, a)}{\partial \log z_i}.$$

Then, after introducing the dual variable y (the effective Fayet-Iliopoulos parameter), we arrive at the A -polynomial:

$$(38) \quad \log y = \frac{\partial \widetilde{\mathcal{W}}_{\text{eff}}(x, a)}{\partial \log x} \quad \Leftrightarrow \quad A_K(x, y, a) = 0.$$

4. RELATIONS WITH ENUMERATIVE GEOMETRY

In this section we look at F_K invariants and Alexander polynomials from the point of view of the enumerative geometry in the spirit of large N duality. For comparison with invariants already discussed, we point out that counts of holomorphic curves naturally give invariants in the *fully unreduced* normalisation corresponding to $\bar{P}(K; a, q)$, see Sections 3.1 and 5.1. To get results in the reduced normalisation, one divides by the curve count (or equivalently unreduced invariant) of the unknot.

4.1. Curve counts. We start from the deformed conifold T^*S^3 and two Lagrangians: the knot conormal L_K and the knot complement M_K , which both have the Legendrian conormal $\Lambda_K \subset ST^*S^3$ as ideal boundary. We shift L_K off of the zero-section S^3 along the closed non-vanishing form $d\theta$ that generates $H^1(L_K) = H^1(S^1 \times \mathbb{R}^2)$. We shift M_K similarly along a closed form β that generates $H^1(M_K) = \mathbb{R}$. We take this form to agree with the form $d\mu$ that is dual to the meridian circle on the boundary of a tubular neighbourhood of the knot. If K is fibered, then we can find a non-zero such form β , otherwise not.

We want to count (generalised) holomorphic curves with boundary on L_K or M_K . There are two ways to do this for L_K , either we can consider L_K as a Lagrangian submanifold in the resolved conifold X or we can use a sufficiently SFT-stretched almost complex structure on T^*S^3 for which all curves leave a neighbourhood of the zero section, see [ES19, Section 2.5]. The resulting counts (and in fact the curves) are the same. For M_K the second approach still works: after stretching M_K intersects a neighbourhood of S^3 in T^*S^3 in a finite collection of cotangent fibers. Then, possible curves in the inside region (near S^3) have boundaries on these fibers and positive punctures at Reeb chords corresponding to geodesics connecting them. The dimension of such a curve is

$$\dim = \sum_j (\text{index}(\gamma_j) + 1) \geq 2,$$

where the sum runs over positive punctures of the curve, γ_j is the Reeb chord at the puncture, and $\text{index}(\gamma_j)$ the Morse index of the corresponding geodesic. It follows that no such curve can appear after stretching, since the outside part would then have negative index. This means that there is a curve count also for M_K . As we will discuss below, although this curve count is well defined and invariant, when intersections between S^3 and M_K cannot be removed, it is only one point in a space of curve counts that also takes into account certain punctured curves. The present discussion applies to the more involved curve counts of punctured curves as well.

Logarithms of the variables x and y correspond to cycles in Λ_K : $\log x$ corresponds to the meridian and $\log y$ to the longitude. Then $\log x$ is homologous to zero in L_K and generates $H_1(M_K)$. On the other hand, $\log y$ is homologous to zero in M_K and generates $H_1(L_K)$. We write

$$(39) \quad \Psi_K(y, a, g_s) = \sum_{r \geq 0} \bar{P}_r(K; a, q = e^{g_s}) y^{-r} = \exp(p_K(y, a, g_s))$$

for the generating function counting disconnected generalised holomorphic curves on L_K . Here a curve that goes r times around $\log y$ contributes $wa^{\deg} g_s^{-\chi}$ to the coefficient of y^{-r} , where g_s is the string coupling constant, χ is the Euler characteristic of the curve, w its rational weight, and \deg its degree in $H_2(T^*S^3 \setminus S^3)$ after capping off. The logarithm $p_K(y, a, g_s)$ is the corresponding count of connected curves and we have

$$(40) \quad p_K(y, a, g_s) = g_s^{-1} W_K(y, a) + W_K^0(y, a) + g_s W_K^1(y, a) + \dots,$$

where W_K is the disk potential and W_K^k counts curves of Euler characteristic $\chi = -k$.

For M_K we have analogously the generating function of disconnected curves:

$$(41) \quad \Phi_K(x, a, g_s) = \sum_{r \geq 0} V_r(a, e^{g_s}) x^r = \exp(v_K(x, a, g_s)),$$

where

$$(42) \quad v_K(x, a, g_s) = g_s^{-1} U_K(x, a) + U_K^0(x, a) + g_s U_K^1(x, a) + \dots$$

is the corresponding count of connected curves. In analogy to (40), U_K counts disks and U_K^k counts curves of Euler characteristic $\chi = -k$.

4.2. Curve counts at infinity. As explained in [EN18], there is a similar count of holomorphic curves at infinity, with boundary on $\Lambda_K \times \mathbb{R}$. Consider such curve with one positive degree one chord and the rest degree zero chords. Recording degree zero punctures at positive infinity by variables α_j and negative infinity by dual differential operators $g_s \partial_{\alpha_j}$ (see Figure 1), we count curves at infinity, which leads to an operator \mathbf{H} on the D-module with generators $\alpha_j, \partial_{\alpha_j}, \hat{x}, \hat{y}$. We write $f_K = \sum_I f_{K,I} \alpha_I$ for the count of curves with positive punctures in

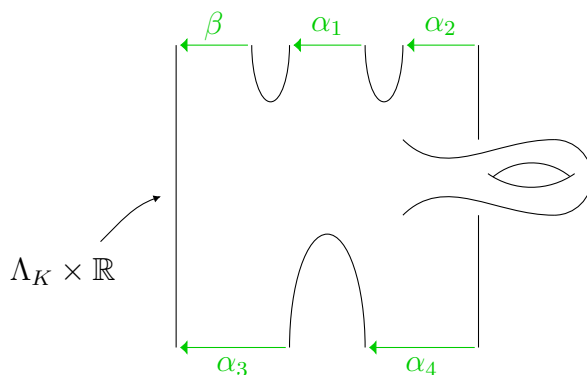


FIGURE 1. A curve at infinity with positive degree one chord β , positive degree zero chords α_1 and α_2 , and negative degree zero chords α_3 and α_4 contributes $g_s^3 \alpha_1 \alpha_2 \partial_{\alpha_3} \partial_{\alpha_4}$ to \mathbf{H} .

the monomials $\alpha_I = \alpha_{i_1} \dots \alpha_{i_k}$. Then our old potential is $f_{K,0}$ (i.e., $f_{K,0} = p_K$ for L_K and $f_{K,0} = v_K$ for M_K) and we have

$$(43) \quad e^{-f_K} \mathbf{H} e^{f_K} = 0,$$

since the left hand side counts ends of a 1-dimensional moduli space. Eliminating $\alpha_j, \partial_{\alpha_j}$ from this equation, we find

$$(44) \quad \hat{A}_K(\hat{x}, \hat{y}, a) e^{f_{K,0}} = 0.$$

4.3. Annulus counts and the Alexander polynomial. In [DE20], counts of holomorphic annuli stretching from M_K to S^3 are considered and it is shown that the Alexander polynomial and A -polynomial are related in the following way:⁶

$$(45) \quad \Delta_K(x) = (1-x) \exp \left(\int \frac{\partial_{\log a} A_K}{\partial_{\log y} A_K} \Big|_{y=1, a=1} d \log x \right).$$

Turning on a and using the parameterisation $y = e^{-\frac{\partial U_K}{\partial \log x}}$, we find that the right hand side is the $a \rightarrow 1$ limit of

$$(1-x) \exp \left(-\frac{\partial U_K}{\partial \log a}(x) \right).$$

We will now consider the counterpart of this equation for all orders in g_s . Fix $a = a_0$ and consider the unnormalised expectation value of the operator $e^{Ng_s \partial_{\log a}}$:

$$(46) \quad \Theta_N(x, a_0, q) = e^{Ng_s \partial_{\log a}} \exp(v_K(x, a_0, g_s)) = \exp(v_K(x, q^N a_0, g_s)).$$

It then follows that $\Theta_N(x, a_0, q)$ satisfies the recursion:

$$(47) \quad \hat{A}_K(\hat{x}, \hat{y}, q^N a_0, q) \Theta_N(x, a_0, q) = 0.$$

Then taking $a_0 \rightarrow 1$ we get

$$(48) \quad \hat{A}_K(\hat{x}, \hat{y}, q^N, q) \Theta_N(x, 1, q) = 0.$$

In other words, $\Theta_N(x, 1, q)$ satisfies the $SU(N)$ -coloured HOMFLY-PT recursion and (after adjusting the normalisation) we can identify it with the F_K invariant

$$(49) \quad \frac{\Delta_K(x)}{(1-x)^N} \cdot \Theta_N(x, 1, q) = F_K^{SU(N), sym}(x, q).$$

Finally, consider the classical limit, corresponding to $g_s \rightarrow 0$:

$$(50) \quad \langle e^{Ng_s \partial_{\log a}} \rangle = e^{-v} e^{Ng_s \partial_{\log a}} e^v, \\ e^{-v} \left(1 + N \frac{\partial U_K}{\partial \log a} + \frac{1}{2} \left(N \frac{\partial U_K}{\partial \log a} \right)^2 + \dots \right) e^v + O(g_s) = e^{N \frac{\partial U_K}{\partial \log a}} + O(g_s).$$

Taking the $a_0 \rightarrow 1$ limit, we get

$$\left(\frac{1-x}{\Delta_K(x)} \right)^N,$$

⁶Strictly speaking, here A_K denotes the augmentation polynomial, which is however conjectured to coincide with the A -polynomial. For details and checks of the conjecture see [AV12, FGS13, FGSS13, AENV14, GKS16, KS16, AMMM14]

and consequently

$$\left(\frac{1}{\Delta_K(x)}\right)^{N-1}$$

for the normalised version in agreement with $\lim_{q \rightarrow 1} F_K^{SU(N),sym}(x, q)$ given by (8).

4.4. Geometric definitions of disk potentials and wave functions. In this section we discuss geometric properties of the curve counts involved in defining the wave function (solutions to the operator equation $\hat{A}_K \Phi_K(x) = 0$) and their semiclassical analogues (solutions to the equation $A_K = 0$ of the form $\log y = -\frac{\partial U_K}{\partial \log x}$, where $U_K = \sum_r c_r x^r$).

In general one expects the wave function to be a count of all (generalised) disconnected holomorphic curves with boundary in M_K . Moreover, the count of disconnected curves is given by the exponential of the count of connected curves, see equations (41-42). When K is fibered, there exists a non-vanishing 1-form on M_K . We use this 1-form to shift $M_K \subset T^*S^3$ off of the zero section S^3 and we consider the curve counts above either in a sufficiently SFT-stretched almost complex structure on T^*S^3 or in the resolved conifold.

When K is not fibered, there is no closed 1-form on M_K without zeros. It is straightforward to arrange that all zeros are critical points of index 1 or 2 (i.e. there are no local maxima or minima). In this case the appropriate form of SFT-stretching leaves the cotangent fiber in T^*S^3 at each critical point. When applying SFT-stretching, curves on the outside may have punctures that end at Reeb chords stretching between fibers. Something similar happens with closed geodesics when stretching around manifolds other than S^3 , see [ES19]. We describe how these curves could be taken into account. We consider first the simpler case of the disk potential, disregarding higher genus curves.

Write ξ_1, \dots, ξ_r and η_1, \dots, η_r for the index 1 and 2 critical points in M_K , respectively. We write ξ_j and η_j also for the corresponding cotangent fibers after stretching and $\partial\xi_j$ and $\partial\eta_j$ for their Legendrian boundary spheres, see Figure 2. Fixing capping paths in the Lagrangian,

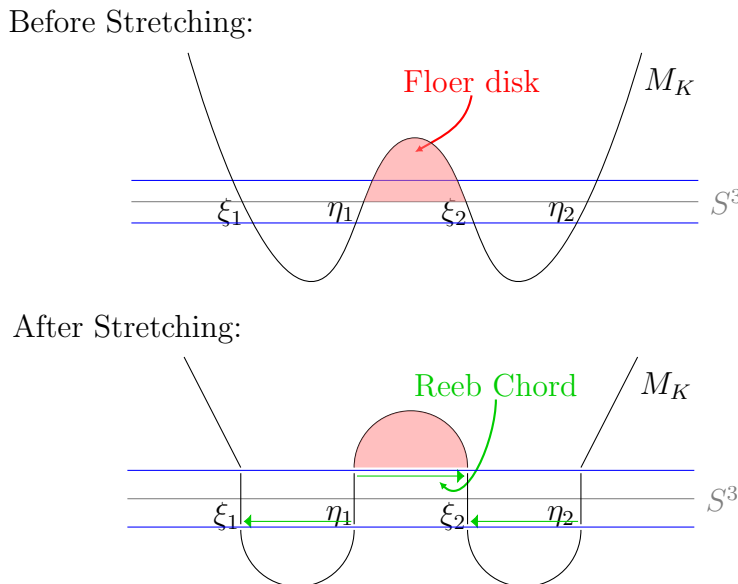


FIGURE 2. The effect of SFT-stretching in the presence of intersections with the zero-section.

there is a natural grading on the Reeb chords which equals the negative of the dimension of a disk with boundary in M_K and negative puncture at the Reeb chord. Then the gradings are as follows:

$$\begin{cases} \partial\eta_j \rightarrow \partial\xi_k & \text{chord, grading} = 0, \\ \partial\eta_j \rightarrow \partial\eta_k & \text{chord, grading} = 1, \\ \partial\xi_j \rightarrow \partial\xi_k & \text{chord, grading} = 1, \\ \partial\xi_j \rightarrow \partial\eta_k & \text{chord, grading} = 2. \end{cases}$$

In the disk potential U_K we would now like to count not only actual closed disks but also disks with negative punctures at the degree 0 chords. Let α_{ij} denote the degree 0 chords, β_{ij} and γ_{ij} the degree 1 chords, and ϵ_{ij} the degree 2 chords. Here β_{ii} and γ_{ii} are formal length zero chords associated with boundaries of bounding chains. We next note that such counts cannot be invariant under deformations for the following reason.

In a generic 1-parameter family there might appear isolated instances where there is a disk σ of dimension -1 . Such disk has negative punctures at one β_{ij} or γ_{ij} chord and the rest at α_{ij} -chords. At such an instance, the count of disks changes by gluing to σ a disk with positive puncture at the degree 1 chord and positive and negative punctures at degree 0 chords, where all positive punctures are capped off by disks with corresponding negative punctures, see Figure 3. This indicates that one should count the disks not with coefficients in the α_{ij} , but rather in an augmentation of the differential graded algebra with differential given by disks with positive degree 0 punctures capped off, see Figure 3. Note that this algebra in degree 0 has linearised homology which is a torsion $x^{\pm 1}$ -module. Its augmentation variety therefore defines α_{ij} as a function of a and x . This variety may have many branches and correspondingly we get several disk potentials.

This raises the question which branch is the right one to give the Alexander polynomial, according to the formula (45). Although it is not easy to characterise that branch concretely, [DE20] shows that such a branch exists as follows. The Alexander polynomial is given by a product

$$(51) \quad \Delta_K(x) = (1 - x) \exp(B_K(x)) \det(D_K(x)),$$

where $D_K(x) = D_{K,0} + O(x)$ is the differential on the Morse-Novikov complex of M_K viewed as an $\mathbb{C}[x^{\pm 1}]$ -module, and $B_K(x)$ is the count of holomorphic annuli stretching between M_K and S^3 . We point out that the coefficient $\Delta_{K,0}$ of the leading term in the Alexander polynomial is

$$\Delta_{K,0} = \det(D_{K,0}),$$

which is equal to 1 for fibered knots, see the discussion in Section 5.6. Here the left hand side remains constant under deformations. On the right hand side, at non-generic instances factors may move from the second to the third factor or in the opposite direction. For sufficiently stretched almost complex structures there can be no further moving of factors and therefore $\log \det(D_K(x))$ should be the contribution to $\frac{\partial U_K}{\partial \log a}$ coming from disks with additional punctures.

As an illustration of the differential graded algebra at the negative end, we consider the basic case when we add two canceling critical points of the shifting 1-form. This leads to an algebra with one chord α of degree 0, two chords β and γ of degree 1, and one chord ϵ of degree 2. The relevant part of the differential is related to the Floer disk that cancels the two critical points. After stretching, this Floer disk gives a disk with a negative puncture at α and homologically trivial boundary. In analogy with ordinary disks, one expects that all

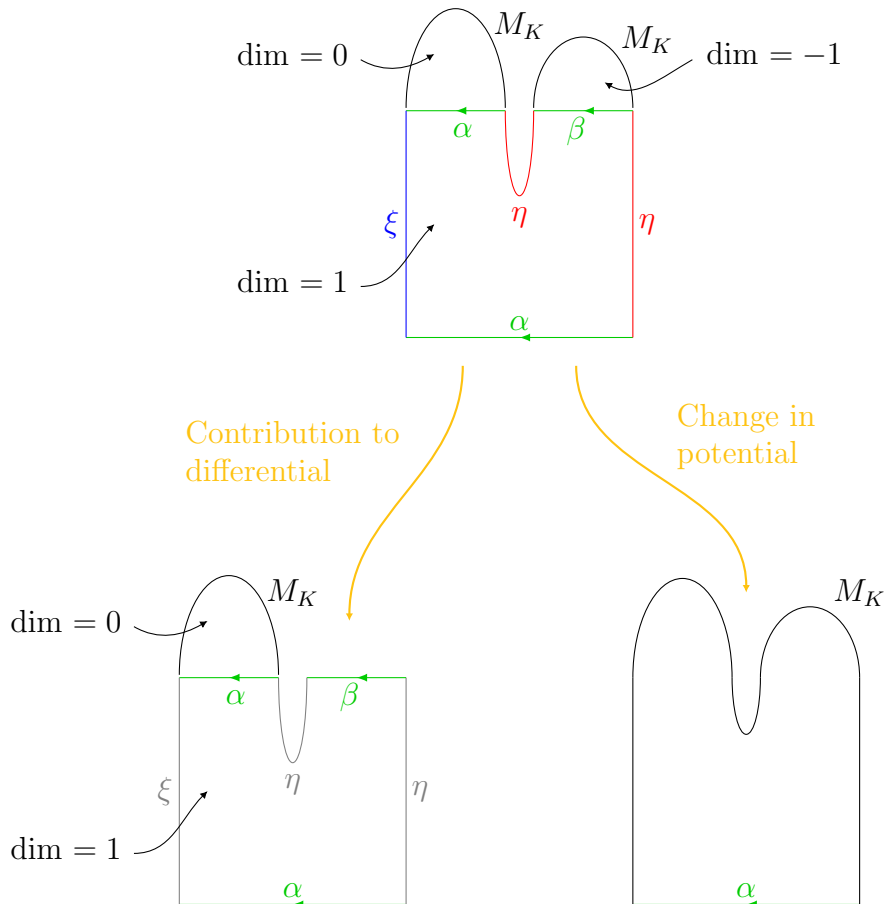


FIGURE 3. Counts of disks with negative punctures changes at instances where there are (-1) -disks. In order to get an invariant, we use augmentations of a differential graded algebra with differential that counts the $(+1)$ -part attached to the (-1) -disk.

its multiple covers contribute to the differential, and taking the puncture into account one gets the count

$$\sum_{k=1}^{\infty} \alpha^k = \frac{\alpha}{1 - \alpha}.$$

An augmentation must vanish on the image of the differential and the augmentation variety is then given by the equation

$$(52) \quad 0 = \frac{\alpha}{1 - \alpha} \quad \text{or} \quad \alpha = 0.$$

It follows that the disk potential remains unchanged, as expected.

In the higher genus case, one should upgrade the differential graded algebra at the negative end just described to an SFT structure. More precisely, capping off with all genus curves instead of only disks, one finds an operator \mathbf{H} that counts curves with one positive puncture at β_{ij} -chord and other punctures at α_{ij} -chords. We require that

$$(53) \quad e^{-f_\alpha} \mathbf{H} e^{f_\alpha} = 0.$$

Eliminating α_{ij} and $\partial\alpha_{ij}$, we get $\alpha_{ij} = \alpha_{ij}(y, q)$ and possibly non-unique wave functions corresponding to different solutions.

5. AN a -DEFORMATION OF F_K

In this section we present our main results on a -deformed F_K invariants – analogues of HOMFLY-PT polynomials for 3-manifolds with the topology of the knot complement. For simplicity we will usually refer only to the knot K , having in mind that the corresponding 3-manifold is $M_K = S^3 \setminus K$.

5.1. Weyl symmetry, normalisations, and the unknot. A few remarks on the conventions we use are in order. We use the *positive expansion* of F_K , meaning that we express F_K as a power series in x expanded around $x = 0$. To get the *negative expansion*, the power series expanded around $x = \infty$ we can simply use the Weyl symmetry

$$(54) \quad F_K(x^{-1}, a, q) = F_K(a^{-1}x, a, q).$$

The *balanced expansion* such as the one in (21) is simply the average of the positive and negative expansions.

In the literature one can find three different normalisations corresponding to different values of F_K for the unknot:

- In *reduced* normalisation it is simply set to be 1. This normalisation is present in the majority of papers on HOMFLY-PT, superpolynomials, and A -polynomials, e.g. [DGR06, AGSF12, FGS13, FGSS13, NRZS12].
- In *unreduced* normalisation it is equal to (60) – the numerator of the full unknot factor. This convention is dominant in the growing literature on F_K invariants, e.g. [GM19, Par20a, Par20b, GHN⁺20], usually combined with the balanced expansion.
- In *fully unreduced* normalisation it is equal to the full unknot factor (57). This normalisation is natural in the context of enumerative invariants and can be found in [OV00, AENV14, EN18, EKL20b, EKL20a, ES19, DE20]. In the literature this normalisation is usually called just “unreduced”, but since we join different perspectives, we have to distinguish it from the one discussed in the previous point.

We present our results mostly in the reduced normalisation. In case of the geometric considerations in Sections 4 and 7 we analyse curve counts leading to fully unreduced normalisation and explain how to obtain the reduced one. Conjecture 1 is formulated in the reduced normalisation except for (4) which should be compared in the unreduced normalisation.

In order to familiarise the reader with normalisations, we present the results for the unknot in all of them. The a -deformed F_K invariant can be obtained from the natural HOMFLY-PT polynomial in representation S^r . It is given by [FGS13]

$$(55) \quad \bar{P}_r(0_1; a, q) = a^{-\frac{r}{2}} q^{\frac{r}{2}} \frac{(a; q)_r}{(q; q)_r} = a^{-\frac{r}{2}} q^{\frac{r}{2}} \frac{(a; q)_\infty (q^{r+1}; q)_\infty}{(aq^r; q)_\infty (q; q)_\infty},$$

where

$$(56) \quad (z; q)_n = \prod_{i=1}^{n-1} (1 - zq^i)$$

is the q -Pochhammer symbol. The expression (26) corresponds to $r = 1$.

After performing the substitution $q^r = x$, we obtain the F_K invariant in the fully unreduced normalisation:

$$(57) \quad F_{0_1}^{\text{full.unred.}}(x, a, q) = a^{-\frac{\log x}{2\hbar}} x^{\frac{1}{2}} \frac{(a; q)_\infty (xq; q)_\infty}{(xa; q)_\infty (q; q)_\infty} = x^{-\frac{\log a}{2\hbar} + \frac{1}{2}} \frac{(xq; q)_{\frac{\log a}{\hbar} - 1}}{(q; q)_{\frac{\log a}{\hbar} - 1}}.$$

For $a = q^N$ it reduces to

$$(58) \quad F_{0_1}^{\text{full.unred.}}(x, q^N, q) = x^{-\frac{N-1}{2}} \frac{(xq; q)_{N-1}}{(q; q)_{N-1}},$$

which for $SU(2)$ gives

$$(59) \quad F_{0_1}^{\text{full.unred.}}(x, q^2, q) = \frac{(xq)^{1/2} - (xq)^{-1/2}}{q^{1/2} - q^{-1/2}}.$$

We obtain the F_K invariant in the unreduced normalisation by dropping the prefactors:

$$(60) \quad F_{0_1}^{\text{unred.}}(x, a, q) = \frac{x^{-\frac{\log a}{2\hbar} + \frac{1}{2}} \frac{(a; q)_\infty (xq; q)_\infty}{(xa; q)_\infty (q; q)_\infty}}{x^{-\frac{\log a}{2\hbar} + \frac{1}{2}} \frac{(a; q)_\infty}{(q; q)_\infty}} = \frac{(xq; q)_\infty}{(xa; q)_\infty} = (xq; q)_{\frac{\log a}{\hbar} - 1}.$$

Substituting $a = q^N$ and $a = q^2$, we get

$$(61) \quad F_{0_1}^{\text{unred.}}(x, q^N, q) = (xq; q)_{N-1}, \quad F_{0_1}^{\text{unred.}}(x, q^2, q) = 1 - xq,$$

so we can see that this unknot factor is appropriate for the positive expansion that we use in the paper. The balanced expansion requires keeping the prefactor $-x^{-\frac{\log a}{2\hbar} + \frac{1}{2}}$, which for $SU(2)$ leads to symmetric expression $(xq)^{1/2} - (xq)^{-1/2}$, being the numerator of (59). Taking into account the fact that here $x = q^r$ for S^r , whereas in [GM19] $x = q^n$ for S^{n-1} , one can recognise the familiar factor $x^{1/2} - x^{-1/2}$ corresponding to switching between reduced and unreduced $F_K^{SU(2)}$.

Finally, the reduced F_K invariant is simply set to 1. Following (26), one can also say that it is obtained by the division by the full unknot factor:

$$(62) \quad F_{0_1}^{\text{red.}}(x, a, q) = \frac{x^{-\frac{\log a}{2\hbar} + \frac{1}{2}} \frac{(a; q)_\infty (xq; q)_\infty}{(xa; q)_\infty (q; q)_\infty}}{F_{0_1}^{\text{full.unred.}}(x, a, q)} = 1.$$

Since the unreduced normalisation is absent in the literature on the A -polynomials, let us analyse the recursion $\hat{A}_{0_1}^{\text{unred.}}(\hat{x}, \hat{y}, a, q) F_{0_1}^{\text{unred.}}(x, a, q) = 0$. We have

$$(63) \quad \hat{A}_{0_1}^{\text{unred.}}(\hat{x}, \hat{y}, a, q) = (1 - a\hat{x}) - (1 - q\hat{x})\hat{y},$$

which agrees with the quantum a -deformed A -polynomial from [FGS13] after taking into account dropping the prefactor and the conventional difference $\hat{x}_{\text{FGS}} = \hat{x}q$.

On the other hand, the semiclassical limit of F_K reproduces the twisted superpotential of the 3d $\mathcal{N} = 2$ theory associated to the unknot complement and analysed in [FGS13]:

$$(64) \quad F_{0_1}^{\text{unred.}}(x, a, q) \xrightarrow{\hbar \rightarrow 0} \exp \left[\frac{1}{\hbar} \widetilde{\mathcal{W}}(x, a) + O(\hbar^0) \right],$$

$$\widetilde{\mathcal{W}}(x, a) = \text{Li}_2(x) - \text{Li}_2(ax).$$

Introducing the variable y dual to x we obtain

$$(65) \quad \log y = \frac{\partial \widetilde{\mathcal{W}}(x, a)}{\partial \log x} = \log(1 - ax) - \log(1 - x).$$

In compliance with Section 3.3, this equation is equivalent to the zero locus of the A -polynomial:

$$(66) \quad A_{0_1}^{\text{unred.}}(x, y, a) = 1 - ax - y + xy,$$

which is a classical limit of (63).

5.2. Explicit formula for $(2, 2p + 1)$ torus knots and proof of Theorem 1. Similarly to the unknot case, we can construct the a -deformed F_K invariants for $(2, 2p + 1)$ torus knots basing on the symmetrically coloured HOMFLY-PT polynomials. They are given by [FGSS13]

$$(67) \quad P_r(T^{(2, 2p+1)}; a, q) = \sum_{0 \leq k_p \leq \dots \leq k_1 \leq k_0 = r} a^{pr} q^{-pr} q^{(2r+1)(k_1+k_2+\dots+k_p) - \sum_{i=1}^p k_{i-1}k_i} \\ \times \frac{(q^r; q^{-1})_{k_1} (aq^{-1}; q)_{k_1}}{(q; q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix},$$

where we use the q -binomial

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

Changing the summation to infinity and substituting $q^r = x$ leads to

$$(68) \quad \sum_{0 \leq k_p \leq \dots \leq k_1} x^{p(\frac{\log a}{h} - 1)} x^{2(k_1+\dots+k_p) - k_1} q^{(k_1+k_2+\dots+k_p) - \sum_{i=2}^p k_{i-1}k_i} \\ \times \frac{(aq^{-1}; q)_{k_1} (x; q^{-1})_{k_1}}{(q; q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}.$$

For simplicity we will omit the prefactor $x^{p(\frac{\log a}{h} - 1)}$, but note this needs to be included when applying Weyl symmetry (54). Summing up, the closed form expression for the a -deformed F_K invariant for arbitrary $(2, 2p + 1)$ torus knot is given by

$$(69) \quad F_{T(2, 2p+1)}(x, a, q) = \sum_{0 \leq k_p \leq \dots \leq k_1} x^{2(k_1+\dots+k_p) - k_1} q^{(k_1+k_2+\dots+k_p) - \sum_{i=2}^p k_{i-1}k_i} \\ \times \frac{(aq^{-1}; q)_{k_1} (x; q^{-1})_{k_1}}{(q; q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}.$$

Now we will show that this formula exhibits all properties of Conjecture 2.

Proof of Theorem 1. Constructing $F_{T(2, 2p+1)}(x, a, q)$ from the HOMFLY-PT polynomials automatically guarantees properties (4-5) and (9).

The Alexander polynomial of $(2, 2p + 1)$ torus knot is given by

$$\Delta_{T(2, 2p+1)}(x) = \frac{1 + x^{2p+1}}{1 + x} = 1 - x + x^2 - \dots + x^{2p} = 1 - x(1 - x)(1 + x^2 + \dots + x^{2(p-1)}).$$

When we put $a = 1$ in (69), the q -Pochhammer $(q^{-1}; q)_{k_1}$ is non-zero only for $k_1 = 0, 1$. Hence all k_i must be 0 or 1 and the summation contains exactly $p + 1$ terms, where the i -th term

has $k_j = 1$ for $j \leq i$ and $k_j = 0$ for $j > i$. Simple computations yield 1 for $i = 0$ and

$$x^{2i-1} q^{i-(i-1)} \frac{(1-q^{-1})(1-x)}{(1-q)} = -x^{2i-1}(1-x)$$

for $i > 0$. Therefore

$$(70) \quad F_{T(2,2p+1)}(x, 1, q) = 1 - x(1-x) \sum_{i=1}^p x^{2i-2} = \Delta_{T(2,2p+1)}(x),$$

proving property (6).

Property (7) follows immediately from observing that $F_{T(2,2p+1)}(x, q, q)$ contains the q -Pochhammer $(1; q)_{k_1}$, which is non-zero only for $k_1 = 0$ in which case the whole expression is equal to 1.

In order to show (8), we start with two simple combinatorial identities:

$$\begin{aligned} \left(\frac{1}{1-x}\right)^N &= \sum_{i=0}^{\infty} \binom{N+i-1}{i} x^i \\ (1+x+\dots+x^{p-1})^{k_1} &= \sum_{0 \leq k_p \leq \dots \leq k_2 \leq k_1} x^{(k_2+\dots+k_p)} \binom{k_1}{k_2} \dots \binom{k_{p-1}}{k_p} \end{aligned}$$

These both follow from counting arguments. The first is the number of ways to divide i balls into N buckets and the second is essentially the multinomial theorem with a simple change of variables. Combining them, we expand

$$\begin{aligned} \left(\frac{1}{\Delta_{T(2,2p+1)}(x)}\right)^{N-1} &= \left(\frac{1}{(1-x(1-x)(1+x^2+\dots+x^{2(p-1)}))}\right)^{N-1} \\ &= \sum_{k_1} \binom{N-2+k_1}{k_1} x^{k_1} (1-x)_1^k (1+x^2+\dots+x^{2(p-1)})^{k_1} \\ (71) \quad &= \sum_{0 \leq k_p \leq \dots \leq k_1} \binom{N-2+k_1}{k_1} (1-x)^{k_1} x^{2(k_1+\dots+k_p)-k_1} \binom{k_1}{k_2} \dots \binom{k_{p-1}}{k_p} \\ &= \lim_{q \rightarrow 1} F_{T(2,2p+1)}(x, q^N, q), \end{aligned}$$

which completes the proof. \square

5.3. Example – trefoil knot. As an example, let us analyse the simplest $(2, 2p+1)$ torus knot – the trefoil. For $p = 1$, equation (69) reduces to

$$(72) \quad F_{3_1}(x, a, q) = \sum_{k=0}^{\infty} (xq)^k \frac{(x; q^{-1})_k (aq^{-1}; q)_k}{(q; q)_k},$$

where we omitted the prefactor $x^{\frac{\log a}{h}-1}$. In the known results for particular N , such as [GM19, Par20a], this prefactor reduces to x^{N-1} and is kept explicit.

The function $F_{3_1}(x, a, q)$ is annihilated by the quantum a -deformed A-polynomial

$$(73) \quad \widehat{A}_{3_1}(\widehat{x}, \widehat{y}, a, q) = a_0 + a_1 \widehat{y} + a_2 \widehat{y}^2,$$

where⁷

$$\begin{aligned} a_0 &= q^4 \hat{x}^3 (q \hat{x} - 1) (1 - a q^3 \hat{x}^2), \\ a_1 &= - (1 - a q^2 \hat{x}^2) (1 + q^4 \hat{x}^2 - a q \hat{x}^2 + a^2 q^4 \hat{x}^4 + q^2 \hat{x} (-1 + q \hat{x} - a q \hat{x} - a q^2 \hat{x}^2)), \\ a_2 &= (1 - a q \hat{x}) (1 - a q \hat{x}^2). \end{aligned}$$

Similarly to the unknot case, the semiclassical limit of $F_{3_1}(x, a, q)$ reproduces the twisted superpotential of $T[M_{3_1}]$, the trefoil complement theory studied in [FGS13]:

$$(74) \quad \begin{aligned} F_{3_1}(x, a, q) &\xrightarrow{\hbar \rightarrow 0} \exp \int \frac{dz}{z} \left[\frac{1}{\hbar} \widetilde{\mathcal{W}}(z, x, a) + O(\hbar^0) \right], \\ \widetilde{\mathcal{W}}(z, x, a) &= \log x \log z - \text{Li}_2(x) + \text{Li}_2(xz^{-1}) + \text{Li}_2(a) - \text{Li}_2(az) + \text{Li}_2(z). \end{aligned}$$

Extremalisation with respect to z (z_0 denotes the extremal value) and introduction of the variable y dual to x leads to equations

$$(75) \quad \begin{cases} 1 = \frac{x(1-xz_0^{-1})(1-az_0)}{(1-z_0)}, \\ y = \frac{z_0(1-x)}{1-xz_0^{-1}}. \end{cases}$$

Eliminating z_0 , we obtain the A -polynomial

$$(76) \quad A_{3_1}(x, y, a) = (x-1)x^3 - (1-x+2(1-a)x^2 - ax^3 + a^2x^4)y + (1-ax)y^2,$$

which is the classical limit of (73).

We would like to compare our results with F_K invariants for $SU(2)$ case from [GM19] and $SU(N)$ case from [Par20a]. For this, we need to prepare F_K for the right-handed trefoil (which is the mirror of left-handed trefoil we presented above) in the unreduced normalisation, rescaling x , keeping the prefactor x^{N-1} , and using the balanced expansion (see Section 5.1 and remarks before it). Let us go through all this conventional changes step by step for the $SU(2)$ case; for $SU(N)$ the method is completely analogous.

We start from the formula (72) with the prefactor $x^{N-1} = x$ and set $a = q^2$ to obtain

$$(77) \quad \sum_{k=0}^{\infty} x(xq)^k (x; q^{-1})_k.$$

In order to get the formula for the right-hand trefoil, we have to perform the change of variables

$$x \mapsto x^{-1}, \quad q \mapsto q^{-1}$$

and use the Weyl symmetry

$$x^{-1} \mapsto ax = q^2x,$$

which gives

$$(78) \quad \sum_{k=0}^{\infty} xq^2(xq)^k (xq^2; q)_k.$$

Then we have to switch from $x = q^r$ for S^r to $x = q^n$ for S^{n-1} used in [GM19], which corresponds to $x \mapsto x/q$. Performing this transformation and expanding in x we get

$$qx + qx^2 + q(1-q)x^3 + q(1-q-q^2)x^4 + q(1-q-q^2+q^5)x^5 + \dots$$

⁷Formula (73) differs from [FGS13] by the rescaling of \hat{x} by q mentioned earlier, and of \hat{y} by a/q due to the omitted prefactor.

Since [GM19] uses the unreduced normalisation, we multiply by $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$, which leaves us with the series

$$-qx^{\frac{1}{2}} + q^2x^{\frac{5}{2}} + q^3x^{\frac{7}{2}} - q^6x^{\frac{11}{2}} - q^8x^{\frac{13}{2}} + \dots$$

From here we simply need to switch to the balanced expansion, which – thanks to the $SU(2)$ Weyl symmetry – means replacing $x^n \mapsto \frac{1}{2}(x^n - x^{-n})$, and we exactly recover (q, x) -series from [GM19, eq.(114)]:

$$(79) \quad F_{3_1}^{SU(2)}(x, q) = -\frac{1}{2} \left[q(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) - q^2(x^{\frac{5}{2}} - x^{-\frac{5}{2}}) - q^3(x^{\frac{7}{2}} - x^{-\frac{7}{2}}) \right. \\ \left. + q^6(x^{\frac{11}{2}} - x^{-\frac{11}{2}}) + q^8(x^{\frac{13}{2}} - x^{-\frac{13}{2}}) + \dots \right].$$

Following the same procedure, we find a perfect agreement with the results of [Par20a], presented below for $N = 3, 4$ in the unreduced normalisation and using balanced expansion. Here \cong denotes equality up to sign and a multiplication by a monomial q^d for some $d \in \mathbb{Q}$.

$$(80) \quad F_{3_1}^{\text{sym,unred},SU(3)}(x, q) \cong \frac{1}{2} \left[(q^{1/2}x + q^{-1/2}x^{-1})(1) \right. \\ \left. + (q^{3/2}x^3 + q^{-3/2}x^{-3})(-q - q^2) \right. \\ \left. + (q^2x^4 + q^{-2}x^{-4})(-q^{5/2} - q^{7/2}) \right. \\ \left. + (q^{5/2}x^5 + q^{-5/2}x^{-5})(q^3) \right. \\ \left. + (q^3x^6 + q^{-3}x^{-6})(q^{9/2} + q^{11/2} + q^{13/2} + q^{15/2}) + \dots \right],$$

$$(81) \quad F_{3_1}^{\text{sym,unred},SU(4)}(x, q) \cong \frac{1}{2} \left[(q^{3/2}x^{3/2} - q^{-3/2}x^{-3/2})(1) \right. \\ \left. + (q^{7/2}x^{7/2} - q^{-7/2}x^{-7/2})(-q - q^2 - q^3) \right. \\ \left. + (q^{9/2}x^{9/2} - q^{-9/2}x^{-9/2})(-q^3 - q^4 - q^5) \right. \\ \left. + (q^{11/2}x^{11/2} - q^{-11/2}x^{-11/2})(q^3 + q^4 + q^5) \right. \\ \left. + (q^{13/2}x^{13/2} - q^{-13/2}x^{-13/2})(q^5 + 2q^6 + 2q^7 + 2q^8 + q^9 + q^{10}) + \dots \right].$$

5.4. Solving A -polynomial recursion and proof of Theorem 2. Unfortunately, the strategy of substituting $q^r = x$ does not work for all knots. For example, the coloured HOMFLY-PT polynomial for the figure-eight knot is given in [FGS13] as

$$(82) \quad P_r(4_1; a, q) = \sum_{k=0}^{\infty} (-1)^k a^{-k} q^{-k(k-3)/2} \frac{(aq^{-1}; q)_k}{(q; q)_k} (q^{-r}; q)_k (aq^r; q)_k.$$

Making the substitution $q^r = x$, we get

$$\sum_{k=0}^{\infty} (-1)^k a^{-k} q^{-k(k-3)/2} \frac{(aq^{-1}; q)_k}{(q; q)_k} (x^{-1}; q)_k (ax; q)_k,$$

but this expression does not give a well-defined power series in x .

In a case like this we can construct a candidate for F_K invariant by solving the recursion given by the quantum a -deformed A -polynomial. The uniqueness of the solution is ensured by the following:

Lemma 1. *Suppose that the quantum A -polynomial is properly normalised so that we expect a solution $F_K(x, a, q)$ to the equation*

$$(83) \quad \widehat{A}_K(\widehat{x}, \widehat{y}, a, q)F_K(x, a, q) = 0$$

of the form

$$(84) \quad F_K(x, a, q) = f_0 + f_1(a, q)x + f_2(a, q)x^2 + \dots$$

with $f_0 = 1$. Let us write the quantum A -polynomial as follows:

$$\widehat{A}_K(\widehat{x}, \widehat{y}, a, q) = \sum_{j=0}^d \alpha_j(\widehat{y}, a, q)x^j.$$

Then the equation (83) has a unique solution of the form (84) if and only if

$$(85) \quad \alpha_0(1, a, q) = 0 \quad \text{and} \quad \alpha_0(q^j, a, q) \neq 0$$

for every $j \in \mathbb{Z}_+$. If these conditions are satisfied, then the unique solution is given recursively by

$$(86) \quad f_j(a, q) = -\frac{1}{\alpha_0(q^j, a, q)} \sum_{k=0}^{j-1} \alpha_{j-k}(q^k, a, q)f_k(a, q)$$

for each $j \in \mathbb{Z}_+$.

Proof. The statement follows from simple computations. □

In order to show properties (6-8), we will need also the following:

Lemma 2. *Let $F(x, a, q)$ be a polynomial annihilated by a relation*

$$(87) \quad \widehat{C}(\widehat{x}, \widehat{y}, a, q) = \sum_{i=0}^n c_i(\widehat{x}, a, q)\widehat{y}^i,$$

where each c_i is a rational function. Then \widehat{C} factors as

$$(88) \quad \widehat{C}(\widehat{x}, \widehat{y}, a, q) = \widehat{Q}(\widehat{x}, \widehat{y}, a, q) \left(\widehat{y} - \frac{F(qx, a, q)}{F(x, a, q)} \right).$$

As a brief comment, note that by carefully clearing denominators, this is essentially equivalent to the case where the coefficients c_i are polynomials and the factor has the form $F(x, a, q)\widehat{y} - F(qx, a, q)$.

Proof. The proof is constructive. Define $\widehat{Q}_n(\widehat{x}, \widehat{y}, a, q) = c_n(\widehat{x}, a, q)\widehat{y}^{n-1}(\widehat{y} - \frac{F(qx, a, q)}{F(x, a, q)})$ and $\widehat{C}_{n-1} = \widehat{C} - \widehat{Q}_n$. Then \widehat{C}_{n-1} is an $n - 1$ degree polynomial in \widehat{y} which annihilates $F(x, a, q)$. Repeating this procedure, we end up with a zero degree polynomial $\widehat{C}_0(\widehat{x}, a, q)$ satisfying $\widehat{C}_0(x, a, q)F(x, a, q) = 0$ which, implies $\widehat{C}_0(x, a, q) = 0$. Working backwards, we reconstruct \widehat{C} as $\widehat{Q}_1 + \dots + \widehat{Q}_n$ which manifestly factors as $\widehat{Q}(\widehat{x}, \widehat{y}, a, q)(\widehat{y} - \frac{F(qx, a, q)}{F(x, a, q)})$. □

Since all functions involved in the proof are rational, it is equally valid in the case where instead of $\widehat{C}_0(x, a, q)F(x, a, q) = 0$, our condition is $\widehat{C}_0(q^r, a, q)F(q^r, a, q) = 0$ for all $r \in \mathbb{N}$.

Having Lemma 1 and 2, we are ready to show that our candidate for F_K invariant is well-defined and exhibits properties (6-8) of Conjecture 1:

Proof of Theorem 2. Since the abelian branch of the considered a -deformed A -polynomial \hat{A}_K is non-degenerate, conditions (85) are satisfied. Following Lemma 1, this ensures that $F_K(x, a, q)$ given by (84, 86) is a unique solution of the q -difference equation (83).

By construction, $F_K(x, a, q)$ solves the recurrence relation given by $\hat{A}_K(\hat{x}, \hat{y}, a, q)$ with leading coefficient 1, which automatically ensures (5). On the other hand, the abelian branch is the unique solution whose specialisations $a = q^N$ are non-singular in the $q \rightarrow 1$ limit. Therefore, if the A -polynomial factors as $\hat{A}_K(\hat{x}, \hat{y}, q^N, q) = \hat{Q}(\hat{x}, \hat{y}, q) \left(\hat{y} - \frac{F(qx, q)}{F(x, q)} \right)$ with F being a polynomial with constant term equal to 1, then $F_K(x, q^N, q) = F(x, q)$.

Consider the $a = 1$ limits of coloured HOMFLY-PT polynomial:

$$(89) \quad \bar{\mathcal{P}}_r(K; 1, q) = \Delta_K(q^r),$$

Then Lemma 2 shows that $\hat{y} - \frac{\Delta_K(qx)}{\Delta_K(x)}$ can be factored out from $\hat{A}_K(\hat{x}, \hat{y}, 1, q)$, which means that (6) is satisfied.⁸

Similarly, the $a = q$ limit

$$(90) \quad \bar{\mathcal{P}}_r(K; q, q) = 1$$

combined with Lemma 2 implies that $\hat{y} - 1$ can be factored out from $\hat{A}_K(\hat{x}, \hat{y}, q, q)$, which ensures property (7).

Finally, the property (8) follows directly from the reasoning presented in Section 4.3 \square

5.5. Example – figure-eight knot. Let us see the application of Theorem 2 on the example of the figure-eight knot. The quantum a -deformed A -polynomial is given by [FGS13]:

$$(91) \quad \hat{A}_{4_1}(\hat{x}, \hat{y}, a, q) = a_0 + a_1 \hat{y} + a_2 \hat{y}^2 + a_3 \hat{y}^3$$

where⁹

$$\begin{aligned} a_0 &= -\frac{(1 - q\hat{x})(1 - q^2\hat{x})(1 - aq^4\hat{x}^2)(1 - aq^5\hat{x}^2)}{q(1 - aq\hat{x})(1 - aq^2\hat{x})(1 - aq\hat{x}^2)(1 - aq^2\hat{x}^2)}, \\ a_1 &= \frac{(1 - q^2\hat{x})(1 - aq^5\hat{x}^2)}{q^4\hat{x}^2(1 - aq\hat{x})(1 - aq^2\hat{x})(1 - aq\hat{x}^2)} \\ &\quad \times (-1 + 2q^2\hat{x} + aq(1 - q - q^2 + q^3)\hat{x}^2 + aq^3(-1 + q + q^2 - q^3)\hat{x}^3 - 2a^2q^5\hat{x}^4 + a^2q^7\hat{x}^5), \\ a_2 &= \frac{(1 - aq^4\hat{x}^2)}{q^4\hat{x}^2(1 - aq^2\hat{x})(1 - aq^2\hat{x}^2)} \\ &\quad \times (1 - 2aq\hat{x} - aq^2(1 - q)^2(1 + q)\hat{x}^2 + a^2q^3(1 - q - q^2 + q^3)\hat{x}^3 + 2a^2q^7\hat{x}^4 - a^3q^8\hat{x}^5), \\ a_3 &= \frac{a^2}{q}. \end{aligned}$$

⁸Note that there is a minor technicality when $\Delta_K(x)$ is not monic as in that case we should get $\frac{\Delta_K(x)}{a}$ where a is the leading coefficient. However this case doesn't occur as when $\Delta_K(x)$ is not monic the abelian branch should be degenerate.

⁹Again we rescale \hat{x} by q , \hat{y} by a/q and remove common factors of a, q . The A -polynomial we use corresponds to the reduced normalisation.

We can see that $\alpha_0 = y - 1$, so there exists a unique solution – a candidate for the F_K invariant. We take the ansatz¹⁰

$$F_{4_1}(x, a, q) = \sum_{k=0}^{\infty} f_k(a, q)x^k,$$

and then use \widehat{A}_{4_1} to recursively solve for the f_k . This leaves us with one free variable, f_0 , which we set to be 1 for now.¹¹ The first few terms are

$$(92) \quad \begin{aligned} f_0 &= 1, \\ f_1 &= -\frac{3(a-q)}{(1-q)}, \\ f_2 &= -\frac{(a-q)(1-2a+6q-6aq+2q^2-aq^2)}{(1-q)(1-q^2)}, \\ f_3 &= -\frac{(1-a)(a-q)(2+3q-5aq+11q^2-6aq^2+6q^3-11aq^3+5q^4-3aq^4-2aq^5)}{(1-q)(1-q^2)(1-q^3)}. \end{aligned}$$

This looks relatively arcane but there is another form which makes slightly clearer. Denoting

$$(a)^{(n)} = \prod_{i=1}^n \frac{(a-q^i)}{(1-q^i)} = \frac{a^n (a^{-1}q; q)_n}{(q; q)_n}$$

we find that we can write our functions as follows:

$$(93) \quad \begin{aligned} f_0 &= (a)^{(0)} = 1, \\ f_1 &= -3(a)^{(1)}, \\ f_2 &= -(1+6q+q^2)(a)^{(1)} + (2+6q+q^2)(a)^2, \\ f_3 &= -\left(2+3q+11q^2+3q^3+2q^4\right)(a)^{(1)} + \left(2+8q+17q^2+14q^3+5q^4+2q^5\right)(a)^{(2)} \\ &\quad - \left(5q+6q^2+11q^3+3q^4+2q^5\right)(a)^{(3)}. \end{aligned}$$

5.6. 5_2 knot. Unfortunately, there are many knots for which (85) is not satisfied (it seems that non-fiberedness is correlated with the non-uniqueness). Twist knots K_n with $|n| > 1$ are good examples. For K_n , $\alpha_0(y, a, q)$ has a factor of $\prod_{j=0}^{|n|-1} (y - q^j)$, meaning that the first n coefficients, f_0, f_1, \dots, f_{n-1} are free parameters and cannot be determined by just solving the recursion. Below we study the example of $K_2 = 5_2$ in detail to illustrate this point.

We find that, although the first two coefficients f_0, f_1 seem to be free parameters, we can do better than that; in particular, by imposing non-singularity condition for $F_K(x, e^{N\hbar}, e^{\hbar})$ in the limit $\hbar \rightarrow 0$, the term f_1 is determined by f_0 . Schematically,

$$(\widehat{A}_K F_K = 0) + (\text{the non-singularity condition}) \Rightarrow \text{unique solution, up to an overall factor.}$$

The 5_2 knot is just an example, and we conjecture that this procedure works for every knot. In terms of the curve of the A -polynomial $\{A_K = 0\}$, this means that we expect a unique wave function once a branch of the curve near $x = -\infty$ has been specified. We also note that the appearance of many branches of the curve of the A -polynomial indicates that the form $\log x d(\log y)$ is singular along the curve.

¹⁰The prefactor $x^{\frac{\log a}{\hbar} - 1}$ is omitted.

¹¹In general, we cannot simply set $f_0 = 1$; it should be determined by means other than recursion.

The (reduced) quantum a -deformed A -polynomial for the 5_2 knot can be found in [NRZS12, FGSS13]. After aligning with the conventions we are using, it is given by

$$(94) \quad \widehat{A}_{5_2}(\hat{x}, \hat{y}, a, q) = a_0 + a_1 \hat{y} + a_2 \hat{y}^2 + a_3 \hat{y}^3 + a_4 \hat{y}^4,$$

where

$$\begin{aligned} a_0 &= -aq^{12}\hat{x}^7(q\hat{x} - 1)(q^2\hat{x} - 1)(q^3\hat{x} - 1)(aq^5\hat{x}^2 - 1)(aq^6\hat{x}^2 - 1)(aq^7\hat{x}^2 - 1), \\ a_1 &= q^6\hat{x}^2(q^2\hat{x} - 1)(q^3\hat{x} - 1)(aq^2\hat{x}^2 - 1)(aq^6\hat{x}^2 - 1)(aq^7\hat{x}^2 - 1)(a^3q^9\hat{x}^6 + a^3q^8\hat{x}^6 \\ &\quad - 3a^2q^8\hat{x}^5 - a^2q^8\hat{x}^4 - a^2q^7\hat{x}^5 - a^2q^7\hat{x}^4 + a^2q^6\hat{x}^4 - a^2q^4\hat{x}^4 - a^2q^3\hat{x}^4 + aq^8\hat{x}^4 + aq^7\hat{x}^4 \\ &\quad + 2a q^7\hat{x}^3 + aq^6\hat{x}^4 - aq^5\hat{x}^3 - aq^5\hat{x}^2 - aq^4\hat{x}^3 + 2a q^3\hat{x}^3 + aq^3\hat{x}^2 + aq^2\hat{x}^2 - aq\hat{x}^2 + q^4\hat{x}^2 \\ &\quad - 2q^2\hat{x} + 1), \\ a_2 &= -q(q^3\hat{x} - 1)(aq\hat{x} - 1)(aq\hat{x}^2 - 1)(aq^4\hat{x}^2 - 1)(aq^7\hat{x}^2 - 1)(a^4q^{16}\hat{x}^8 - 2a^3q^{15}\hat{x}^7 \\ &\quad - a^3q^{14}\hat{x}^7 - a^3q^{14}\hat{x}^6 - a^3q^{13}\hat{x}^6 - a^3q^{11}\hat{x}^6 - a^3q^{10}\hat{x}^6 + 2a^2q^{14}\hat{x}^6 + 3a^2q^{13}\hat{x}^6 + 2a^2q^{13}\hat{x}^5 \\ &\quad + a^2q^{12}\hat{x}^5 - 2a^2q^{11}\hat{x}^5 + a^2q^{11}\hat{x}^4 - a^2q^{10}\hat{x}^5 + 2a^2q^9\hat{x}^5 + a^2q^9\hat{x}^4 + a^2q^8\hat{x}^5 + 2a^2q^8\hat{x}^4 \\ &\quad + a^2q^7\hat{x}^4 + a^2q^5\hat{x}^4 - aq^{13}\hat{x}^5 - aq^{12}\hat{x}^5 - 2a q^{12}\hat{x}^4 - aq^{11}\hat{x}^5 - 2a q^{11}\hat{x}^4 + aq^{10}\hat{x}^4 + 2a q^9\hat{x}^4 \\ &\quad + 2a q^9\hat{x}^3 - aq^8\hat{x}^4 + aq^8\hat{x}^3 - 2a q^7\hat{x}^4 - 2a q^7\hat{x}^3 - aq^6\hat{x}^3 - aq^6\hat{x}^2 + 2a q^5\hat{x}^3 - aq^5\hat{x}^2 \\ &\quad + aq^4\hat{x}^3 - aq^3\hat{x}^2 - aq^2\hat{x}^2 - q^9\hat{x}^3 - q^8\hat{x}^3 - q^7\hat{x}^3 + 2q^6\hat{x}^2 + 3q^5\hat{x}^2 - 2q^3\hat{x} - q^2\hat{x} + 1), \\ a_3 &= (aq\hat{x} - 1)(aq^2\hat{x} - 1)(aq\hat{x}^2 - 1)(aq^2\hat{x}^2 - 1)(aq^6\hat{x}^2 - 1)(a^3q^{16}\hat{x}^6 - 2a^2q^{14}\hat{x}^5 - a^2q^{13}\hat{x}^4 \\ &\quad + a^2q^{11}\hat{x}^4 + a^2q^{10}\hat{x}^4 - a^2q^9\hat{x}^4 + aq^{12}\hat{x}^4 + 2a q^{11}\hat{x}^3 - aq^9\hat{x}^3 - aq^8\hat{x}^3 - aq^8\hat{x}^2 + 2a q^7\hat{x}^3 \\ &\quad - aq^7\hat{x}^2 + aq^6\hat{x}^2 - aq^4\hat{x}^2 - aq^3\hat{x}^2 + q^8\hat{x}^2 + q^7\hat{x}^2 + q^6\hat{x}^2 - 3q^4\hat{x} - q^3\hat{x} + q + 1), \\ a_4 &= (aq\hat{x} - 1)(aq^2\hat{x} - 1)(aq^3\hat{x} - 1)(aq\hat{x}^2 - 1)(aq^2\hat{x}^2 - 1)(aq^3\hat{x}^2 - 1). \end{aligned}$$

Solving the recursion, we find that the first two coefficients, f_0 and f_1 , determine all the others. That is,¹²

$$(95) \quad F_{5_2}(x, a, q) = \sum_{j \geq 0} f_j(a, q)x^j,$$

where f_j with $j \geq 2$ are $\mathbb{Q}(a, q)$ -linear combinations of f_0 and f_1 . For instance,

$$f_2 = -\frac{(a^2 + a(q^2 - 4q - 2) + q(-q^2 + 3q + 2))}{(q - 1)^2(q + 1)}f_0 + \frac{(aq + a - 3q - 1)}{q^2 - 1}f_1.$$

Although at this point it may seem like f_0 and f_1 are free parameters, we have additional conditions to impose, namely that $F_K(x, q^N, q)$ is non-singular in the semiclassical limit $\hbar \rightarrow 0$. Note that this non-singularity condition is weaker than imposing the explicit limit (8), but as we will see, powers of Alexander polynomial automatically pop up just from this non-singularity condition. This non-singularity property imposes lots of conditions on the perturbative coefficients of f_0 and f_1 , and in particular it determines the entire

¹²The prefactor $x^{\frac{\log a}{\hbar} - 1}$ is omitted.

perturbative series of the ratio f_1/f_0 :

$$\begin{aligned}
 \frac{f_1}{f_0}(a = e^{N\hbar}, q = e^{\hbar}) &= \frac{3}{2}(N-1) \\
 &+ \frac{5}{8}N(N-1)\hbar \\
 &+ \frac{3}{16}N(N-1)(2N-1)\frac{\hbar^2}{2!} \\
 &+ \frac{1}{64}N(N-1)(17N^2 - 17N - 3)\frac{\hbar^3}{3!} \\
 &+ \frac{1}{320}N(N-1)(66N^3 - 99N^2 + 11N + 41)\frac{\hbar^4}{4!} \\
 &\vdots
 \end{aligned}
 \tag{96}$$

Plugging this back in and setting $\lim_{q \rightarrow 1} f_0(q^N, q) = 2^{1-N}$, we see that the expected properties (6)-(8) in Conjecture 1 hold! That is, when $a = 1$, we have $f_0(q^0, q) = 2$, $f_1(q^0, q) = -3$, and

$$F_{5_2}(x, q^0, q) = 2 - 3x + 2x^2,
 \tag{97}$$

whereas for $a = q$ we get

$$F_{5_2}(x, q^1, q) = 1.
 \tag{98}$$

Similarly, when $a = q^N$ (N not necessarily an integer), we have

$$\lim_{q \rightarrow 1} F_{5_2}(x, q^N, q) = (2 - 3x + 2x^2)^{1-N}.
 \tag{99}$$

Expressing the series in terms of \hbar and $N\hbar$ instead, we get

$$\begin{aligned}
 (q-1)\frac{f_1}{f_0}(a, q) \Big|_{a=e^{N\hbar}, q=e^{\hbar}} &= \left(3(N\hbar/2) + 5\frac{(N\hbar/2)^2}{2!} + 9\frac{(N\hbar/2)^3}{3!} + 17\frac{(N\hbar/2)^4}{4!} + \dots \right) \\
 &+ \left(-\frac{3}{2} + \frac{1}{4}(N\hbar/2) + \frac{1}{4}\frac{(N\hbar/2)^2}{2!} + \frac{1}{4}\frac{(N\hbar/2)^3}{3!} + \dots \right) \hbar \\
 &+ \left(-\frac{3}{2} + \frac{1}{8}(N\hbar/2) + \frac{1}{8}\frac{(N\hbar/2)^3}{3!} + \frac{1}{8}\frac{(N\hbar/2)^5}{5!} + \dots \right) \frac{\hbar^2}{2!} \\
 &+ \left(-\frac{3}{2} + \frac{5}{32}(N\hbar/2) + \frac{1}{16}\frac{(N\hbar/2)^2}{2!} + \frac{13}{32}\frac{(N\hbar/2)^3}{3!} + \dots \right) \frac{\hbar^3}{3!} \\
 &\vdots
 \end{aligned}
 \tag{100}$$

It is an interesting problem to re-sum these perturbative series into expressions in a and q .

6. A t -DEFORMATION OF F_K

In the previous section we found that the a -deformed F_K invariants for $(2, 2p+1)$ torus knots can be derived from the HOMFLY-PT polynomials. Since the latter admit a categorification [Kho07], which was quite unexpected in the mathematical literature and emerged from physics [GSV05, DGR06], we can follow this path and propose (a, t) -deformed F_K invariants based on the superpolynomials. These (a, t) -deformed invariants can also be computed term by term using the super- A -polynomial introduced in [AGSF12, FGS13].

6.1. Explicit formula for $(2, 2p + 1)$ torus knots and proof of Theorem 3. Let us construct the explicit formula for the (a, t) -deformed F_K invariants for $(2, 2p + 1)$ torus knots basing on the superpolynomials.

We start from the unknot corresponding to $p = 0$. The formula for the fully unreduced superpolynomial is given by [FGS13]

$$(101) \quad \bar{\mathcal{P}}_r(0_1; a, q, t) = a^{-\frac{r}{2}} q^{\frac{r}{2}} (-t)^{-\frac{3r}{2}} \frac{(-at^3; q)_r}{(q; q)_r} = \frac{(-at^3; q)_\infty (q^{r+1}; q)_\infty}{(-aq^r t^3; q)_\infty (q; q)_\infty}.$$

After substituting $q^r = x$ and dropping the prefactors, we obtain the (a, t) -deformed F_K invariant in the unreduced normalisation:

$$(102) \quad F_{0_1}^{\text{unred.}}(x, a, q, t) = \frac{(xq; q)_\infty}{(-xat^3; q)_\infty}.$$

It is annihilated by the quantum super- A -polynomial

$$(103) \quad \hat{A}_{0_1}(\hat{x}, \hat{y}, a, q, t) = (1 + at^3 \hat{x}) - (1 - q\hat{x})\hat{y},$$

which, just as in the previous section, agrees with [FGS13] after taking into account the changes of conventions. Moreover, we can see that (102) matches (60) for $t = -1$.

For $(2, 2p + 1)$ torus knots with $p \geq 1$, we use the formula for reduced superpolynomials from [FGSS13]:

$$(104) \quad \mathcal{P}_r(T^{(2, 2p+1)}; a, q, t) = \sum_{0 \leq k_p \leq \dots \leq k_1 \leq k_0 = r} q^{(2r+1)(k_1+k_2+\dots+k_p) - \sum_{i=1}^p k_{i-1}k_i} t^{2(k_1+\dots+k_p)} \\ \times a^{pr} q^{-pr} \frac{(q^r; q^{-1})_{k_1} (-aq^{-1}t; q)_{k_1}}{(q; q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \dots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}.$$

Substituting $q^r = x$ and omitting the prefactor $x^{p(\frac{\log a}{h} - 1)}$ (in analogy to Section 5.2), we obtain the closed form expression for the (a, t) -deformed $F_{T(2, 2p+1)}$ in the reduced normalisation:

$$(105) \quad F_{T(2, 2p+1)}(x, a, q, t) = \sum_{0 \leq k_p \leq \dots \leq k_1} x^{2(k_1+\dots+k_p) - k_1} q^{(k_1+k_2+\dots+k_p) - \sum_{i=2}^p k_{i-1}k_i} t^{2(k_1+\dots+k_p)} \\ \times \frac{(-aq^{-1}t; q)_{k_1} (x; q^{-1})_{k_1}}{(q; q)_{k_1}} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \dots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix}.$$

Now we will show that this formula exhibits all properties of Conjecture 2.

Proof of Theorem 3. Constructing $F_{T(2, 2p+1)}(x, a, q, t)$ from the superpolynomials automatically guarantees properties (13) and (17). One can also easily check that (105) agrees with (69) for $t = -1$, which shows (12).

Moreover, we have

$$(106) \quad F_{T(2, 2p+1)}(x, -t^{-1}, q, t) = \frac{1 - t^2x + t^{2p+2}x^{2p+1} - t^{2p+2}x^{2p+2}}{1 - t^2x^2} \\ = 1 - t^2x(1-x)(1 + (tx)^2 + \dots + (tx)^{2(p-1)}) \\ = \mathcal{P}_{T(2, 2p+1)}(-t^{-1}, x, t) = \Delta_{T(2, 2p+1)}(x, t),$$

which proves property (14).

Property (15) follows immediately from noticing that for $a = -t^{-1}q$ we obtain the q -Pochhammer $(1; q)_{k_1}$, which is non-zero only for $k_1 = 0$ and then the whole summand is equal to 1.

Finally, we have

$$\begin{aligned}
 \left(\frac{1}{\Delta_{T(2,2p+1)}(x,t)} \right)^{N-1} &= \left(\frac{1}{1 - t^2x(1-x)(1 + (tx)^2 + \dots + (tx)^{2(p-1)})} \right)^{N-1} \\
 &= \sum_{k_1 \geq 0} \binom{N-2+k_1}{k_1} t^{2k_1} x^{k_1} (1-x)^{k_1} (1 + (tx)^2 + \dots + (tx)^{2(p-1)})^{k_1} \\
 &= \sum_{0 \leq k_p \leq \dots \leq k_1} \binom{N-2+k_1}{k_1} x^{2(k_1+\dots+k_p)-k_1} t^{2(k_1+\dots+k_p)} \\
 &\quad \times (1-x)^{k_1} \binom{k_1}{k_2} \dots \binom{k_{p-1}}{k_p} \\
 &= \lim_{q \rightarrow 1} F_{T(2,2p+1)}(x, -t^{-1}q^N, q, t),
 \end{aligned}$$

which confirms (16) and completes the proof. \square

6.2. Solving A-polynomial recursion – figure-eight knot. Because of the existence of the super- A -polynomial [FGS13], we have an immediate corollary of Theorem 2:

Corollary 1. *Any function that is a candidate for F_K invariant built order by order following Theorem 2 has a natural t -deformation inherited from the super- A -polynomial.*

Let us see it on the example of the figure-eight knot. From [FGS13], we find the coefficients of the super- A -polynomial $\hat{A}_{4_1}(\hat{x}, \hat{y}, a, q, t) = a_0 + a_1\hat{y} + a_2\hat{y}^2 + a_3\hat{y}^3$ to be:¹³

$$\begin{aligned}
 a_0 &= \frac{t^3(1-q\hat{x})(1-q^2\hat{x})(1+aq^4t^3\hat{x}^2)(1+aq^5t^3\hat{x}^2)}{q(1+aq^3\hat{x})(1+aq^2t^3\hat{x})(1+aq^2t^3\hat{x}^2)(1+aq^3\hat{x}^2)}, \\
 a_1 &= \frac{(1-q^2\hat{x})(1+aq^5t^3\hat{x}^2)}{q^4\hat{x}^2(1+aq^3\hat{x})(1+aq^2t^3\hat{x})(1+aq^3\hat{x}^2)} \\
 &\quad \times \left(1 + q^2t(1-t)\hat{x} + aqt^3(1+q^3+qt+q^2t)\hat{x}^2 - aq^3t^4(q+q^2+t+q^3t)\hat{x}^3 \right. \\
 &\quad \left. + a^2q^5t^6(1-t)\hat{x}^4 - a^2q^7t^8\hat{x}^5 \right), \\
 a_2 &= -\frac{(1+aq^4t^3\hat{x}^2)}{q^4\hat{x}^2(1+aq^2t^3\hat{x})(1+aq^2t^3\hat{x}^2)} \\
 &\quad \times \left(1 + aqt(1-t)\hat{x} + aq^2t^2(q+q^2+t+q^3t)\hat{x}^2 + a^2q^3t^4(1+q^3+qt+q^2t)\hat{x}^3 \right. \\
 &\quad \left. - a^2q^7t^5(1-t)\hat{x}^4 + a^3q^8t^7\hat{x}^5 \right), \\
 a_3 &= -\frac{a^2t^3}{q}.
 \end{aligned} \tag{107}$$

Solving this recurrence relation as before, with the ansatz

$$F_{4_1}(x, a, q, t) = \sum_{k=0}^{\infty} f_k(a, q, t)x^k,$$

¹³In this refined case we rescale \hat{y} by $-at/q$, \hat{x} by q and then we remove common factors of a, q, t

we find that the first few f_k are given by

$$\begin{aligned}
 f_0 &= (-at)^{(0)} = 1, \\
 f_1 &= -(1 - t + t^2)(-at)^{(1)}, \\
 f_2 &= -(-q + qt + q^2t - t^2 - 2qt^2 + t^3 + qt^3 - qt^4)(-at)^{(2)} \\
 &\quad + (-q + qt + q^2t - 2qt^2 + t^3 + qt^3 - qt^4)(-at)^{(1)}, \\
 f_3 &= -q\left(q^2 - q^2t - q^3t - q^4t + t^2 + qt^2 + 2q^2t^2 + q^3t^2 + q^4t^2 - t^3 \right. \\
 &\quad \left. - 2qt^3 - 3q^2t^3 - q^3t^3 - 2t^4 + 2qt^4 + 2q^2t^4 - t^5 - qt^5 - q^2t^5 + q^2t^6\right)(-at)^{(3)} \\
 (108) \quad &+ \left(q^2 + q^3 - q^2t - 2q^3t - 2q^4t - q^5t + qt^2 + 3q^2t^2 + 3q^3t^2 \right. \\
 &\quad \left. + 2q^4t^2 + q^5t^2 - 2qt^3 - 5q^2t^3 - 4q^3t^3 - q^4t^3 + t^4 + 3qt^4 \right. \\
 &\quad \left. + 4q^2t^4 + 2q^3t^4 - t^5 - 2qt^5 - 2q^2t^5 - q^3t^5 + q^2t^6 + q^3t^6\right)(-at)^{(2)} \\
 &- \left(q^2 - q^2t - q^3t - q^4t + 2q^2t^2 + q^3t^2 + q^4t^2 - qt^3 \right. \\
 &\quad \left. - 3q^2t^3 - q^3t^3 + t^4 + qt^4 + 2q^2t^4 - t^5 - qt^5 - q^2t^5 + q^2t^6\right)(-at)^{(1)},
 \end{aligned}$$

where $(-at)^{(n)} = \prod_{i=1}^n \frac{(-at - q^i)}{(1 - q^i)}$. It is easy to see that for $t = -1$ we recover our previous result (93), whereas $a = -t^{-1}$ specialisation is consistent with

$$(109) \quad \Delta_{4_1}(x, t) = \mathcal{P}_{4_1}(-t^{-1}, x, t) = t^{-1}x^{-1} + (-t^{-1} + 1 - t) + tx.$$

6.3. t -deformed ADO polynomials. In a recent work [GHN⁺20], certain connections between \widehat{Z} and non-semisimple modular tensor categories were observed. In particular, in Conjecture 3 of that paper, it was conjectured that ADO polynomials can be obtained as limits of F_K as q approaches roots of unity $\zeta_p = e^{2\pi i/p}$, up to a factor determined by the Alexander polynomial $\Delta_K(x)$. Since certain specialisations of the (a, t) -deformed F_K lead to t -deformed Alexander polynomials $\Delta_K(x, t)$, it is tempting to use them to study certain limits of (a, t) -deformed F_K as q approaches roots of unity. Below we give a t -deformed version of Conjecture 3 of [GHN⁺20], where the role of Alexander polynomial is replaced by that of the t -deformed one.

Conjecture 4. *The limit*

$$(110) \quad \text{ADO}_K(p; x, t) := \Delta_K(x^p, -(-t)^p) \lim_{q \rightarrow e^{2\pi i/p}} F_K(x, -t^{-1}q^2, q, t)$$

is a polynomial, and when $t = -1$, it is the usual p -th ADO polynomial of K for $SU(2)$.

More generally, the limit

$$(111) \quad \text{ADO}_K^{SU(N)}(p; x, t) := \Delta_K(x^p, -(-t)^p)^{N-1} \lim_{q \rightarrow e^{2\pi i/p}} F_K(x, -t^{-1}q^N, q, t)$$

is a polynomial and is a t -deformation of the symmetric version of the p -th ADO polynomial of K for $SU(N)$ (i.e. $P_K^{SU(N)}$ introduced in [GHN⁺20], specialised to $x_1 = qx$, $x_2 = \dots = x_{N-1} = q$).

Using the data we provided in this section, we can explicitly compute the t -deformed ADO polynomials for many knots. Below, in Tables 2 and 3, we summarise the computation for the trefoil and the figure-eight knot. We have written it in a way that the Weyl symmetry is manifest. Note that $\text{ADO}_K(2; x, t)$ is always $\Delta_K(x, t)$.

| p | $\text{ADO}_{3_1}(p; x, t)$ |
|-----|--|
| 1 | 1 |
| 2 | $(-tx) + t^{-1} + (-tx)^{-1}$ |
| 3 | $(t\zeta_3x)^2 + t^{-1}(t\zeta_3x) + (t^{-2} - \zeta_3) + t^{-1}(t\zeta_3x)^{-1} + (t\zeta_3x)^{-2}$ |

 TABLE 2. t -deformed ADO polynomials for the right-handed trefoil knot

| p | $\text{ADO}_{4_1}(p; x, t)$ |
|-----|---|
| 1 | 1 |
| 2 | $-(-tx) + (-t + 1 - t^{-1}) - (-tx)^{-1}$ |
| 3 | $(t\zeta_3x)^2 + (t - 1 + t^{-1})(t\zeta_3x) + (t^2 + \zeta_3t + 2 + \zeta_3^{-1}t^{-1} + t^{-2})$ $+ (t - 1 + t^{-1})(t\zeta_3x)^{-1} + (t\zeta_3x)^{-2}$ |

 TABLE 3. t -deformed ADO polynomials for the figure-eight knot

Computations for the first few values of N suggest that the t -deformed $SU(N)$ -ADO polynomial has the Weyl symmetry

$$(112) \quad \text{ADO}_K^{SU(N)}(p; x^{-1}, t) = \text{ADO}_K^{SU(N)}(p; \zeta_p^{-N} t^{-2} x, t).$$

This in turn suggests the Weyl symmetry of the (a, t) -deformed F_K :

$$F_K(x^{-1}, -t^{-1}a, q, t) = F_K(t^{-2}a^{-1}x, -t^{-1}a, q, t),$$

or simply

$$(113) \quad F_K(x^{-1}, a, q, t) = F_K(-t^{-3}a^{-1}x, a, q, t).$$

The available data suggests that the $SU(N)$ -ADO polynomials fall into patterns depending on $N \pmod p$:

Conjecture 5. *The t -deformed p -th $SU(N)$ -ADO polynomials satisfy the relation*

$$(114) \quad \text{ADO}_K^{SU(N+p)}(p; x, t) = \Delta_K(x^p, -t^p)^{p-1} \text{ADO}_K^{SU(N)}(p; x, t).$$

This, in particular, implies

$$(115) \quad \begin{aligned} \text{ADO}_K^{SU(pN)}(p; x, t) &= \Delta_K(x, t) \Delta_K(x^p, -(-t)^p)^{(p-1)N-1}, \\ \text{ADO}_K^{SU(pN+1)}(p; x, t) &= \Delta_K(x^p, -(-t)^p)^{(p-1)N}. \end{aligned}$$

Finally, we note that it is also possible to make sense of $SU(N)$ -ADO invariant for generic N . This gives rise to a two variable series in x, t with coefficients being functions of N . The first couple of terms in this series for the right-handed trefoil are shown in Table 4.

| p | $(tx)^{(p-1)(N-1)} \text{ADO}_{3_1}^{SU(N)}(p; x, t)$ |
|-----|---|
| 2 | $\left(1 - \frac{1}{2}(1 + (-1)^N)x + \frac{1}{4}(3 + (-1)^N - 2N)x^2 + O(x^3)\right)$ $+ \left(\frac{1+(-1)^N}{2}x^2 - \frac{1+(-1)^N}{4}(-2 + N)x^4 + O(x^6)\right) t^2$ $+ \left(-\frac{3+(-1)^N+2N}{4}x^4 - \frac{1+(-1)^N}{4}(-2 + N)x^5 + O(x^6)\right) t^4 + O(t^6)$ |
| 3 | $\left(1 + \frac{1}{3}((\zeta_3 - 1) + ((\zeta_3^2 - 1))\zeta_3^{2N})x + O(x^2)\right)$ $+ \left(\frac{\zeta_3^N}{3}(1 - \zeta_3 + (1 - \zeta_3^2)\zeta_3^N)x^2 + O(x^3)\right) t^2 + O(t^4)$ |
| 4 | $\left(1 + \frac{1}{2}(i - 1)(1 + i^{N+1})x + \frac{1}{4}(i - 1)(1 - i^N)(i - i^N)x^2 + O(x^3)\right)$ $+ \left(\frac{(-1)^N}{2}(1 - i)(i + i^{3N})x^2 + O(x^3)\right) t^2 + O(t^4)$ |

 TABLE 4. N, t -deformed ADO series for the right-handed trefoil knot

6.4. Physical and geometric meaning of t . Physically, the q -series invariants \widehat{Z} and their variants F_K for knot complements are generating functions of integer BPS invariants, cf. the lower-right corner of Table 1. In particular, they are defined as graded traces over the spaces of BPS states, $\mathcal{H}_{i,j,\beta}^{\text{BPS}}$, so that the latter provide categorification of \widehat{Z} and F_K .

In the same way, the a -dependent invariants studied in this paper encode the graded dimensions of the spaces of BPS states on the *resolved* side of the geometric transition (34):

$$(116) \quad F_K(x, a, q) := \sum_{i,j,\beta_1,\beta_2} (-1)^i q^j a^{\beta_1} x^{\beta_2} \text{rank} \mathcal{H}_{i,j,\beta_1,\beta_2}^{\text{BPS}}(X, L_K)$$

Introducing a new variable t' and replacing $(-1)^i$ on the right-hand side by $(t')^i$, we obtain the Poincaré polynomial of $\mathcal{H}^{\text{BPS}}(X, L_K)$. In the case of knots in S^3 , this gives the Poincaré polynomial of the coloured HOMFLY-PT homology (a.k.a. the coloured superpolynomial) such that, possibly up to a simple change of variables, $t' = t$.

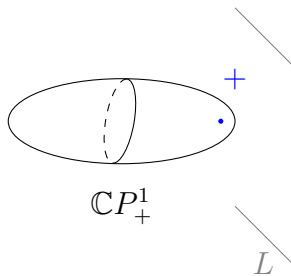
One motivation for the present work is to gain access to \mathcal{H}^{BPS} in the case of 3-manifolds. Note that in this case the Calabi-Yau geometry X on the resolved side (34) depends on the choice of 3-manifold Y . Of course, for $Y = S^3$ with no knots in it, X is just the resolved conifold. Even in this case, the space of BPS states has a very rich structure [GPV17], in particular it has the right structure to produce spaces $\mathcal{H}_{SU(N)}^{\text{BPS}}$ on the deformed side via spectral sequences with differentials d_N . These are the same type of differentials that relate e.g. Khovanov homology and its Lee deformation. For other 3-manifolds the spaces of BPS states on the resolved side are not known explicitly. However, if one can identify $t' = t$ as in the case of knot invariants, then the computation of t -dependent $F_K(x, a, q, t)$ in this paper provides the desired graded Poincaré polynomial of $\mathcal{H}^{\text{BPS}}(M_K)$ on the resolved side. Whether this is true can be checked in a number of ways.

For example, one can ask whether the t -deformation $F_K(x, a, q, t)$ computed here has the right structure to reproduce the finite-rank $SU(N)$ version by taking cohomology with respect to the differentials d_N , as in the case of knots and as in the case of $Y = S^3$. In other words, the differentials d_N relate the spaces of BPS states on the resolved (HOMFLY-PT) side and on the deformed $SU(N)$ side. Moreover, if we know t -deformed \widehat{Z} -invariants on both sides, we can simply check whether the difference is of the form $(1 + t^i q^j a^\beta)(\dots)$ with particular $(i, j, \beta) = \text{deg}(d_N)$ as in the case of knots and $Y = S^3$.

From the point of view of the geometric interpretation of quiver nodes as basic holomorphic disks [EKL20b, EKL20a], it is natural to conjecture that the counterpart of refined Chern-Simons theory has to do with distinguishing the self-linking of the boundary of a basic disk from the 4-chain intersections in its interior. For bare curves (i.e. perturbed curves of positive symplectic area that are embedded) self-linking of the boundary can be traded for 4-chain intersections [ES19]. For basic disks the situation is different, as they should be considered with all their multiple covers. Note here that when counting generalised holomorphic curves induced by degree d covers of the basic disk, the boundary self-linking ℓ contributes quadratically ($q^{\ell d^2}$), whereas the 4-chain intersection c contributes linearly (q^{cd}). From the point of view of knot conormals, one would guess that refined invariants are defined for (possibly singular) special Lagrangians associated to a knot. Candidates for such Lagrangians are covers of the unknot conormal branched along a link.

Here we provide further evidence for how 4-chain intersections contribute to refined curve counting. Consider a basic holomorphic $\mathbb{C}P^1$ in a Calabi-Yau 3-fold and a Lagrangian brane L that moves so as to intersect the basic sphere. Then the basic sphere passes through the Lagrangian, but a new moduli space consisting of a basic disk appears. The difference between the sphere before and after the crossing with L is that one 4-chain intersection changes its sign, see Figure 4. Assume now that the basic disk that appears has no self-linking,

Before crossing:



After crossing:

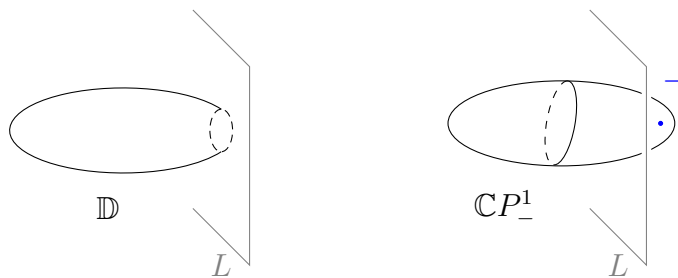


FIGURE 4. Basic holomorphic spheres and disks for refined curve counting.

so that its partition function is

$$\exp \left(- \sum_{d=1}^{\infty} \frac{a^d}{d(q^{d/2} - q^{-d/2})} \right).$$

According to [AS12], the refined partition function of $\mathbb{C}\mathbf{P}^1$ is

$$\exp\left(\sum_{d=1}^{\infty} \frac{a^d}{d(q^{d/2} - q^{-d/2})(t^{d/2} - t^{-d/2})}\right).$$

This then indicates that we should count the contribution to the refined open string by 4-chain intersections, which would then give an invariant refined partition function as follows. Write $\mathbb{C}\mathbf{P}^1_-$ and $\mathbb{C}\mathbf{P}^1_+$ for the sphere before and after the L intersection, respectively, and \mathbb{D} for the disk. Then

$$\begin{aligned} \Psi_{\mathbb{C}\mathbf{P}^1_+} \Psi_{\mathbb{C}\mathbf{P}^1_-}^{-1} &= \exp\left(-\sum_{d=1}^{\infty} \frac{t^{d/2} a^d}{d(q^{d/2} - q^{-d/2})(t^{d/2} - t^{-d/2})} + \sum_{d=1}^{\infty} \frac{t^{-d/2} a^d}{d(q^{d/2} - q^{-d/2})(t^{d/2} - t^{-d/2})}\right) \\ (117) \quad &= \exp\left(-\sum_{d=1}^{\infty} \frac{a^d}{d(q^{d/2} - q^{-d/2})}\right) = \Psi_{\mathbb{D}}. \end{aligned}$$

7. MIRROR KNOTS AND RELATIONS WITH QUANTUM MODULARITY

In this section we consider how F_K -invariants behave under mirroring and find indications of interesting modularity properties. One of the main results of this work is that the geometric realisation of F_K in terms of curve counting as well as the explicit examples we studied all suggest the existence of an a -deformation of two-variable series $F_K(x, q)$ for knot complements. A simple but interesting corollary of this is that for every knot K the a -deformed functions $F_K(x, a, q)$ come in pairs,

$$(118) \quad F_K(x, a, q) \quad \xleftarrow{\text{mirror}} \quad F_{m(K)}(x, a, q),$$

where $m(K)$ denotes the mirror knot. If the knot K is amphichiral, i.e. $K \simeq m(K)$, then the above relation is simply an equality. It becomes very interesting and highly nontrivial, though, when K is *not* amphichiral.¹⁴

In order to explain this simple but important point, let us recall that under $K \mapsto m(K)$ the coloured HOMFLY-PT polynomials behave as

$$(119) \quad P_R(m(K); a, q) = P_R(K; a^{-1}, q^{-1})$$

for any colour R (including the symmetric ones, most relevant to us here). The behaviour is consistent with (and can be derived from) a similar behaviour in Chern-Simons theory under the orientation reversal. Since the entire Chern-Simons functional changes sign under parity (orientation reversal), it has the same effect as changing the sign of the “level” or, equivalently, $q \mapsto q^{-1}$. This is true for any $G = SU(N)$ and, therefore, can be summarised by $(a, q) \mapsto (a^{-1}, q^{-1})$, which is precisely the statement in (119).

Similarly, if we consider Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$ on the knot complement, $S^3 \setminus K$, then the parity (orientation reversal) operation induces¹⁵ an orientation reversal on the boundary torus $T^2 \cong \partial(S^3 \setminus K)$. This tells us that out of two $G_{\mathbb{C}}$ -valued holonomies x and y along 1-cycles of T^2 one should be inverted upon $K \mapsto m(K)$. In particular, when $G_{\mathbb{C}} = SL(2, \mathbb{C})$ this implies the familiar transformation of the A-polynomial.

¹⁴Actually it is interesting for non-fibered amphichiral knots too, as it will give us a q -series which is invariant under $q \leftrightarrow q^{-1}$.

¹⁵One way to see this is to consider a simple example, e.g. a solid torus $\cong S^3 \setminus \text{unknot}$, and realise the parity transformation as a sign change of one of the coordinates along the boundary.

Since the latter is defined only up to overall powers of x and y , and enjoys the Weyl symmetry $A_K(x, y) \sim A_K(x^{-1}, y^{-1})$, it does not matter whether we use $(x, y) \mapsto (x^{-1}, y)$ or $(x, y) \mapsto (x, y^{-1})$ as the effect of the orientation reversal. The standard choice is

$$(120) \quad A_{m(K)}(x, y) = A_K(x^{-1}, y).$$

Note that the transformations (119) and (120) are compatible. Indeed, as we reviewed earlier, introducing a -dependence and quantising the classical curve $A_K(x, y) = 0$ leads to a recursion relation $\hat{A}_K(\hat{x}, \hat{y}, a, q)P_*(K; a, q) = 0$ that encodes colour dependence of the coloured HOMFLY-PT polynomials. According to (119), the recursion relation for the mirror knot should involve q^{-1} in place of q (accompanied by a more straightforward transformation $a \mapsto a^{-1}$). Since $\hat{x}P_r = q^r P_r$ and $\hat{y}P_r = P_{r+1}$, it follows that under $q \mapsto q^{-1}$ we have $\hat{x} \mapsto \hat{x}^{-1}$ and $\hat{y} \mapsto \hat{y}$. Therefore, the (a, q) -deformed version of (120) involves $(\hat{x}, \hat{y}, a, q) \mapsto (\hat{x}^{-1}, \hat{y}, a^{-1}, q^{-1})$ under $K \mapsto m(K)$.

Now we are ready to explain why the relation (118) between the two functions $F_K(x, a, q)$ and $F_{m(K)}(x, a, q)$ is very nontrivial when K is not amphichiral. Since a coefficient of each term $a^{n_a} x^{n_x}$ is supposed to be in $\mathbb{Z}[q^{-1}, q]$, we can not simply take $q \mapsto q^{-1}$. Therefore, from this point of view, $F_K(x, a, q)$ and $F_{m(K)}(x, a, q)$ are very different! For instance, while in this paper we discuss the form of $F_{5_2}(x, a, q)$, it does not tell us directly what the corresponding terms in $F_{m(5_2)}(x, a, q)$ should be.

On the other hand, the relation between $F_K(x, a, q)$ and $F_{m(K)}(x, a, q)$ at the perturbative level is very direct and simple.¹⁶ As usual, let us write $a = e^{N\hbar}$ and $q = e^{\hbar}$, where N can be treated as a parameter (not necessarily integer). In these variables, the mirror transform is $(x, N, \hbar) \mapsto (x^{-1}, N, -\hbar)$. Therefore, to all orders in the perturbative expansion with respect to a and q near 1, we have

$$(121) \quad F_K(x, a, q) = \sum_{i,j \geq 0} \sum_m c_{i,j,m} \hbar^i N^j x^m, \quad F_{m(K)}(x, a, q) = \sum_{i,j \geq 0} \sum_m c_{i,j,m} (-\hbar)^i N^j x^{-m}$$

with the *same* coefficients $c_{i,j,m}$.

The problem, therefore, is to translate a rather simple relation among the perturbative coefficients of $F_K(x, a, q)$ and $F_{m(K)}(x, a, q)$ into a much more sophisticated relation (118). Note that this task would be much easier if one had an *a priori* knowledge about modular properties of $F_K(x, a, q)$ with respect to q , i.e. $\tau = \frac{\hbar}{2\pi i}$. Then, the modular transformation $\tau \mapsto -\frac{1}{\tau}$ could help relating the coefficients of the q -series near the ‘‘cusp’’ $\tau = i\infty$ to the perturbative expansion near $q \simeq 1$ or $\tau \simeq 0$. The tools useful for solving this problem include resurgent analysis [GMP16, AP19] and Rademacher sums [CCF⁺19].

We next consider the counterpart of the above discussion from the point of view of curve counting. Here the substitution $(a, q) \rightarrow (a^{-1}, q^{-1})$ that relates F_K to $F_{m(K)}$ can be derived as follows. Starting from a knot K , we get its mirror knot simply by reversing the orientation of S^3 . Reversing the orientation of S^3 reverses the orientation of the 4-chain of S^3 and as the power of a corresponds to an intersection number between holomorphic curves and the 4-chain, we see that a should be replaced by a^{-1} . Similarly, the orientation on the complement Lagrangian M_K changes and since q counts intersections with its 4-chain (corresponding to the $U(1)$ gauge theory on the single copy of the M_K -brane), we find that q should be replaced by q^{-1} .

¹⁶Since \hat{y} -coefficients of $\hat{A}_K(\hat{x}, \hat{y}, a, q)$ are rational functions of \hat{x} , a , and q , in the quantum A-polynomial one can also simply replace $(\hat{x}, a, q) \mapsto (\hat{x}^{-1}, a^{-1}, q^{-1})$.

We next discuss the coefficients of $a^{n_a}x^{n_x}$ in F_K from the geometric point of view. We start in the case of fibered knots. If K is fibered, then M_K can be made disjoint from S^3 in T^*S^3 and moved to the resolved conifold. In this setting F_K is given by a count of curves with boundary in M_K . Using a perturbation scheme for counting bare curves as in [ES19], the contribution of such a (possibly disconnected) curve u to the count of generalised curves in F_K has the form

$$(122) \quad w(u) (q^{1/2} - q^{-1/2})^{-\chi(u)} a^{d(u)/2} q^{l(u)/2} x^{k(u)},$$

where $w(u)$ is the rational weight of the curve as a point in the moduli space, $\chi(u)$ is the Euler characteristic of u , $d(u)$ the homological degree, $l(u)$ a linking of the boundary of u , and $k(u)$ the boundary degree. This then means that the coefficient of $a^{n_a}x^{n_x}$ lies in $\mathbb{Q}[q^{\pm 1/2}, (q^{1/2} - q^{-1/2})^{-1}]$ which shows that the substitution $q \rightarrow q^{-1}$ taking us from F_K to $F_{m(K)}$ works well. Note that the expression (122) corresponds to the *fully unreduced* normalisation, natural from the geometric perspective. The result in the reduced normalisation can be obtained by dividing by the curve count of the unknot.

If K is not fibered, then – as explained above – we must also take into account contributions from curves with additional negative punctures at Reeb chords α connecting fibers corresponding to intersection points in $M_K \cap S^3 \subset T^*S^3$. The argument above indicates that the functions $\alpha = \alpha(a, q)$ transform via the change of variables $(a, q) \rightarrow (a^{-1}, q^{-1})$ under change of orientation of M_K and S^3 . Given this we would have a similar but somewhat more involved result for the coefficients as follows. The coefficients of $a^{n_a}x^{n_x}$ would take values in $\mathbb{Q}[q^{\pm 1/2}, (q^{1/2} - q^{-1/2})^{-1}] \otimes \mathbb{Q}[[\alpha(a, q)]]$ with change of variables giving coefficients in $\mathbb{Q}[q^{\pm 1/2}, (q^{1/2} - q^{-1/2})^{-1}] \otimes \mathbb{Q}[[\alpha(a^{-1}, q^{-1})]]$. Not much is known about the functions $\alpha(a, q)$ but the examples from Section 5 indicate that they could contain rational powers of a and q .

We also expect this work to offer a new territory for studying quantum modular forms and their generalisations. The notion of quantum modularity, introduced in [Zag10], is about properties of a function defined only at rational numbers, $\mathbb{Q} \subset \mathbb{R}$, on the real axis of the τ -plane. Since in terms of the variable q these are the points on the unit circle, $|q| = 1$, naively it seems that $F_K(x, q^N, q)$ and $F_{m(K)}(x, q^N, q)$ discussed earlier have little to do with quantum modularity because they are defined in the upper half-plane and in the lower half-plane (or, inside and outside the unit disk if we use q instead of τ). However, in all examples (of low rank) that have been studied so far, the connection to quantum modularity was found by studying (regularised) limits to roots of unity in q -variable, i.e. $\tau \rightarrow \tau_0 \in \mathbb{Q}$. Therefore, based on these studies, one might expect that connection to quantum modularity continues to hold more generally.

We can also offer an intuitive reason why one might expect such connection at roots of unity. Conceptually, $F_K(x, q^N, q)$ and $F_{m(K)}(x, q^N, q)$ are quantum group invariants associated with $\mathcal{U}_q(\mathfrak{sl}_N)$ at generic $|q| < 1$. From the theory of quantum groups and from the physical realisation of $F_K(x, q^N, q)$, it is clear that (regularised) limits of such functions should be very interesting and contain rich structure if (and, probably, only if) $q \rightarrow$ root of 1. Moreover, at those points ($\tau \in \mathbb{Q}$) the asymptotic expansions of $F_K(x, q^N, q)$ and $F_{m(K)}(x, q^N, q)$ are expected to be related in a simple way, cf. (121).

Summing up, if the knot K is not amphichiral, the two functions $F_K(x, q^N, q)$ and $F_{m(K)}(x, q^{-N}, q^{-1})$, naturally defined in the upper half-plane and in lower half-plane respectively, are in general quite different and related in a highly nontrivial way, cf. (118). Yet, their asymptotic expansions near rational points on the real axis are related by a very

simple “analytic continuation” (121). This peculiar phenomenon, sometimes called “leaking”, not only provides a function defined on $\tau \in \mathbb{Q}$, but automatically comes equipped with two analytic continuations of this function to the upper and lower half-plane. Furthermore, these two functions are expected to have modular properties of characters of logarithmic vertex algebras [CCF⁺19]. From this perspective, it is perhaps less surprising that limiting values of characters of chiral algebras are related by $SL(2, \mathbb{Z})$ action.

Besides connections to traditional quantum modularity at $a = q^N$, it would be interesting to understand the properties of $F_K(x, a, q)$ itself. It is quite possible that $F_K(x, a, q)$ is also related to characters of (non-unitary) chiral algebras. We hope that exploring this direction can lead to new types of modularity and strengthen the connection between enumerative geometry, quantum algebra, and number theory.

8. FUTURE DIRECTIONS

In this section, we provide a summary of interesting open problems that emerged during our research:

- In this paper we noticed a close relation between a -deformed F_K invariants and HOMFLY-PT polynomials. For $(2, 2p + 1)$ torus knots it led to the closed form expression of $F_K(x, a, q)$ coming from $P_r(K; a, q)$, with $q^r = x$. However, this approach does not work in general: we have seen that for the figure-eight knot such substitution would lead to an ill-defined series containing expansion in both x and x^{-1} . It would be desirable to solve this problem and understand the relation between HOMFLY-PT polynomials and a -deformed F_K invariants in full generality.
- It would be desirable to prove Conjectures 1-2 in full generality and define F_K in a proper mathematical way, showing that it is a topological invariant. Unfortunately, so far this was problematic even in the simplest $SU(2)$ case. There is, however, a definition in $SU(2)$ case for positive braid knots [Par20b], and it should not be hard to generalise it to $SU(N)$. It would be interesting to find the a -deformed F_K for positive braid knots using the same approach, in which the main step is finding the a -deformed R -matrix.
- We demonstrated how to solve the recursion to get a unique solution up to an overall factor. However, we still do not know how to determine this overall factor, namely the first coefficient $f_0(a, q)$, especially for non-fibered knots like 5_2 . Clearly we need a method beyond the recursion. One possible approach is to combine with the expected asymptotic series (9). Another approach would be to use the a -deformed R -matrix, if it exists.
- The close relationship between a -deformed F_K invariants and HOMFLY-PT polynomials suggests that the knots-quivers correspondence [KRSS17, KRSS19] can be generalised to the case of knot complements. The first results presented in [Kuc20] seem to confirm this hypothesis.
- Since knot complements can be glued to give a closed 3-manifold, Dehn surgery of F_K invariants leads to \widehat{Z} invariants [GM19, Par20a]. It would be interesting to find a large N limit and t -deformation of this relation. However, for this we need F_K that takes care of all possible Young diagrams, not just symmetric ones, i.e. allows generic values of all the variables x_i in (2). In the long term one may hope that such developments will be helpful in the categorification of the Witten-Reshetikhin-Turaev invariants.

- As a step toward exploring the relation between t -deformation and categorification of F_K invariants, it would be interesting to study a similar relation between t -deformation and categorification of $\text{ADO}_p(x)$ polynomials that arise as limits of $F_K(x, q)$ at roots of unity. We hope that our computations of t -deformed ADO polynomials in Section 6 will be useful for carrying out this analysis.
- It would be interesting to identify chiral algebras that may have $F_K(x, a, q)$ as their characters. Addressing this is closely related to understanding the modular properties of $F_K(x, a, q)$ as well as its relation to $F_{m(K)}(x, a, q)$. All these questions, that we leave to future work, are intimately interrelated.
- Curiously, both the enumerative perspective discussed here and potential interpretation of $F_K(x, a, q)$ as characters of logarithmic VOAs suggest that $F_K(x, a, q)$ should satisfy q -difference equation with respect to variable a , i.e. q -difference equations where a plays the role similar to that of x and the “shift operator” acts as $a^n \mapsto q^n a^n$.

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